

## A NEGOTIATION PROCESS IMPLEMENTING THE NASH BARGAINING MODEL

### Part I

Tokio Suzuki

Since the introduction of the noncooperative negotiation model in Nash (1953), quite a few authors, including Nash himself, have pointed out problems and proposed more general and realistic models. Nash proved in the same article that the Nash solution, defined in Nash (1950), is the unique equilibrium of a generalized negotiation model using a continuous map as a payoff function. Another noncooperative implementation of the Nash model was made by Rubinstein (1982), in order to get a dynamic character. Since the Nash model is a one-shot game, the negotiations may result in a Pareto inferior outcome. The Rubinstein model is a supergame of the Nash model, where the players offer an outcome alternately until an agreement is reached, and after each round there is some positive probability of a breakdown.

In Section 1 of this article, a general payoff set is introduced, on the basis of Smale (1974a). This set is compact but not necessarily convex. From the results in Nash (1953), the set of equilibria is the union of the Pareto boundary points and the threat point. If the payoff set is convex, and a mixed strategy is allowed, then every point in the payoff set can be obtained as an expected payoff of some equilibrium points. Therefore, the discussion can be restricted to pure strategies, so that the convexity does not play an important role. It is then proved that under some conditions there are a finite number of extended Nash solutions. The proof is based on an infinite dimensional version of the transversality theorem, discussed in Chapter 4 of Abraham and Robbin (1967).

In Section 2, the basic structure of the Rubinstein model is briefly explained. The existence of an equilibrium is then proved, where the general payoff set is used. A further discussion is to be made in another article.

## 1. Nash Solution with General Payoff Sets

A payoff set  $W_0$  is described by  $C^2$  functions  $g_i$ ,  $i=1, \dots, n$ , as follows:

$$W_0 = \{x \in W \mid g_i(x) \geq 0, i=1, \dots, n\}$$

$W = [0,1] \times [0,1]$ . So  $W_0$  is compact in  $R^2$ , but not necessarily convex. The following conditions on  $g_i$  are also assumed.

Assumption 1

- a)  $g_i(0) > 0$ , all  $i$ .
- b) If  $x \in W_0$  and  $B_x$  is the set of  $k \in \{1, \dots, n\}$  such that  $g_k(x) = 0$ , then the vectors  $Dg_k(x)$ ,  $k \in B_x$ , are linearly independent.

a) means that  $0 \in R^2$  is an interior point of  $W_0$ . This point will be defined as the threat point. It follows from b) that  $0 \in R$  is a regular value of  $g_i$ ,  $i=1, \dots, n$ . Since  $W \subset R^2$ ,  $B_x$  can contain at most two elements.  $B_x$  will be replaced by  $B$ , a given subset of  $\{1, \dots, n\}$ .

Let  $C^2(W, R)$  be the space of  $C^2$  maps from  $W$  to  $R$ .  $C^2(W, R)$  is endowed with the  $C^2$  Whitney topology. A neighborhood  $N_h$  of the origin is defined by a continuous map  $h: W \rightarrow R_+$ , where  $R_+$  is the set of strictly positive real numbers, such that:

$f \in C^2(W, R)$  belongs to  $N_h$

if and only if

$$|f(x)| < h(x), \text{ all } x \in W,$$

$$\|Df(x)\| < h(x), \text{ all } x \in W, \text{ and}$$

$$\|D^2f(x)\| < h(x), \text{ all } x \in W.$$

Since  $W$  is compact,  $h$  attains a maximum, so that this topology gives a metric. In fact,  $C^2(W, R)$  with the  $C^2$  Whitney topology becomes a Banach space.

Let  $G$  be the whole collection of  $g = (g_1, \dots, g_n) \in [C^2(W, R)]^n$  satisfying the conditions in Assumption 1. Then it is easily verified that  $G$  is open in  $[C^2(W, R)]^n$ . So  $G$  is a smooth Banach manifold.

The structure of  $W_0$  given by  $g \in G$  is studied in detail. Take any  $B$  and define:

$$b = \{x \in W_0 \mid g_k(x) = 0, \text{ all } k \in B\}$$

From the implicit function theorem,  $g_k^{-1}(0)$ ,  $k \in B$ , becomes a  $C^2$  manifold of dimension one, as long as the inverse image is nonempty. Suppose that  $B$  actually consists of two elements  $k$  and  $k'$ . Then from b) of Assumption 1, the restriction  $g_k|_{g_{k'}^{-1}(0)}$  still has  $0 \in R$  as a regular value. This means that  $b$  is a  $C^2$  manifold, and the dimension is 2 less the number of elements in  $B$ . By using  $g_k|_{g_{k'}^{-1}(0)}$ , it is also easily seen that the tangent space of  $b$  at any  $x \in b$  is given by the following intersection:

$$\bigcap_{k \in B} \text{Ker } Dg_k(x)$$

Definition 1

Take any  $g \in G$  and fix  $B$ . Let  $f: W \rightarrow R$  be  $f: (x_1, x_2) \mapsto x_1 x_2$ .  $x \in b$  is called an extended Nash solution of the game described by  $g$ , with  $B$ , if the following condition is satisfied:

There are nonnegative numbers  $k \in B$ ,  $\lambda$ ,  $\mu_k$ , not all zero such that:

$$\lambda Df(x) + \sum_{k \in B} \mu_k Dg_k(x) = 0$$

This is the first order condition of constrained maximum, as discussed in Chapter 4 of Intriligator (1971).

On the basis of these settings, it can be proved that there is an open and dense subset  $\theta$  of  $G$  such that any  $g \in \theta$  yields a finite number of the extended Nash solutions. Toward this end, the following  $C^1$  map is constructed, where  $g \in G$  and  $B$  are given, and  $\# B$  is the number of elements in  $B$ .

$$\begin{aligned} \psi_{g,B}: S_+(R \times R^{\#B}) \times W &\rightarrow R^{\#B} \times R^2 \\ \text{such that} \\ \psi_{g,B}: (\lambda, \mu, x) &\mapsto \\ &[g_k(x), \lambda \text{ Df}(x) + \sum_{k \in B} \mu_k Dg_k(x)] \end{aligned}$$

The subscript  $k$  ranges over  $B$ , and  $S_+(R \times R^{\#B})$  is the nonnegative part of the unit sphere of dimension  $\# B$ . It is enough to consider a normalized  $(\lambda, \mu_k)$ .

Now consider  $\psi_{g,B}^{-1}(0)$ . Clearly, this is precisely the set of the extended Nash solutions of the game described by  $g$ , with  $B$ . The domain of  $\psi_{g,B}$  is compact, and the dimension of the domain is equal to that of the codomain.

Therefore, if  $\psi_{g,B}$  has  $0 \in R^{\#B} \times R^2$  as a regular value, then the regular value theorem, discussed in Chapter 1 of Hirsch (1976), gives that  $\psi_{g,B}^{-1}(0)$  can contain at most a finite number of elements. Concerning the existence of an extended Nash solution, the classical discussion in Nash (1950) would be enough. A generic property of the finiteness is obtained by the following lemma.

#### Lemma 1

For any  $B$ , there is an open and dense subset  $\theta(B)$  of  $G$  such that each  $\psi_{g,B}^{-1}(0)$ ,  $g \in \theta(B)$ , is a finite set.

#### Proof

Fix  $B$ , and consider the following map:

$$\rho : G \rightarrow C^1 [S_+(R \times R^{\#B}) \times W, R^{\#B} \times R^2]$$

such that

$$\rho : g \longmapsto \psi_{g,B}$$

The codomain of  $\rho$  is endowed with the  $C^1$  Whitney topology, so that it is a Banach space.  $G$  is a smooth Banach manifold. It is then verified that  $\rho$  is a  $C^1$  representation. That is, the evaluation map  $\text{ev}\rho$  is  $C^1$ , where:

$$\text{ev}\rho : G \times S_+(R \times R^{\#B}) \times W \rightarrow R^{\#B} \times R^2$$

such that

$$\text{ev}\rho : (g, \lambda, \mu, x) \longmapsto \rho(g)(\lambda, \mu, x)$$

Then from the discussion in Chapter 4 of Abraham and Robbin (1967), the whole collection of  $g \in G$  such that  $0 \in R^{\#B} \times R^2$  is a regular value of  $\rho g$ , is open in  $G$ . Let  $\theta(B)$  be the set. On the other hand, the  $C^1$  map  $\text{ev}\rho$  has  $0 \in R^{\#B} \times R^2$  as a regular value, and  $G$  is second countable. Therefore, the results in the same chapter of Abraham and Robbin (1967) give that  $\theta(B)$  is also the countable intersection of open and dense subsets of  $G$ . Since  $C^2(W, R)$  endowed with the  $C^2$  Whitney topology satisfies the Baire property, as discussed in Chapter 2 of Hirsch (1976),  $\theta(B)$  is dense in  $G$ .

This completes the proof.

Since the whole collection of  $B$  is finite, the following theorem is obtained by taking the intersection of  $\theta(B)$ .

#### Theorem 1

There is an open and dense subset  $\theta$  of  $G$  such that the set of the extended Nash solutions of the game described by  $g \in \theta$ , is finite.

Remark : From the results in Chapter 4 of Suzuki (1987), a stability theorem can follow. At the moment,  $g \in \theta$  is given and fixed.

## 2. The Rubinstein Model and Some Extension

The negotiation process in Rubinstein (1982), reformulated in Chapter 7 of van Damme (1987), is a multi-stage game with perfect information, where there is some positive probability of a breakdown after each round. This can reflect the costs of bargaining and discounted payoffs.

The number of negotiators is 2, and the breakdown probability  $\delta \in (0,1)$  is given. Each round is described by the following rules, where  $W_0 \subset \mathbb{R}^2$  is the payoff set, and  $N$  is the set of natural numbers:

- 1) In round  $2t-1$ ,  $t \in N$ , player 1 proposes  $x = (x_1, x_2) \in W_0$ . Player 2 can either accept or reject it. If accepted, player  $i$  receives  $x_i$ ,  $i=1, 2$ . Otherwise, the game ends with probability  $\delta$ , yielding the disagreement outcome of  $(0, 0)$ . With probability  $1-\delta$ , the game proceeds to the next round.
- 2) In round  $2t$ ,  $t \in N$ , player 2 offers an element of  $W_0$ . Player 1 can either accept or reject it. The rest is the same as 1).

Suppose that there is a "justified" payoff vector  $x^* = (x_1^*, x_2^*) \in W_0$ , as discussed in Chapter 7 of van Damme (1987). This means that as long as uncertainty is not involved, player  $i$  will never accept any offer less than  $x_i^*$ , and the amount is mutually regarded as reasonable. Furthermore, if  $(x_1^*, (1-\delta)x_2^*)$  and  $((1-\delta)x_1^*, x_2^*)$  belong to the Pareto boundary of  $W_0$ , then the negotiation will reach an agreement in round one, with the payoff  $(x_1^*, (1-\delta)x_2^*)$ . This is easily verified in the following way. Recalling that each player has a utility function of the von Neumann type, player 2 will not reject  $(1-\delta)x_2^*$  in round 1. In round 2, player 1 will accept  $(1-\delta)x_1^*$ .  $(x_1^*, (1-\delta)x_2^*)$  in round 1 is in fact the best reply of player 1 to  $((1-\delta)x_1^*, x_2^*)$  offered by player 2 in round 2. At the same time, the offer  $((1-\delta)x_1^*, x_2^*)$  by player 2 in round 2 is the best reply of player 2

to  $(x_1^*, (1-\delta)x_2^*)$  offered by player 1 in round 1. Under this situation, the bargaining will end in round 1, with the payoff  $(x_1^*, (1-\delta)x_2^*)$ .

These offers are in fact subgame perfect in the corresponding round. The concept of subgame perfectness is explained in Chapter 6 of van Damme (1987).

The existence of  $(x_1^*, x_2^*)$  is proved in the following way. Consider any  $g = (g_1, \dots, g_n) \in G$  such that for any  $x \in g_i^{-1}(0)$ ,  $Dg_i(x)$  is a strictly positive vector in  $\mathbb{R}^2$ ,  $i = 1, \dots, n$ . From the implicit function theorem,  $g_i^{-1}(0)$  in a neighborhood  $N_x$  of  $x$  can be represented as a graph of a  $C^2$  strictly decreasing function on  $\pi(N_x \cap g_i^{-1}(0))$ , where  $\pi: (x_1, x_2) \mapsto x_1$ . Furthermore, it is not difficult to find  $g$  with this property and satisfying that the Pareto boundary of  $W_0$  is given by a graph of a strictly decreasing continuous function  $h$  on  $[0, x_1^0]$ , where  $x_1^0 \leq 1$ , and  $h(x_1^0) = 0$ . If  $n > 1$ ,  $h$  may not be  $C^2$  because of b) of Assumption 1.

## Theorem 2

Suppose that  $g$  satisfies the above properties. Then there is  $(x_1^*, x_2^*) \in W$  such that  $h((1-\delta)x_1^*) = x_2^*$ , and  $(1-\delta)x_2^* = h(x_1^*)$ .

## Proof

This is a straightforward consequence of the discussion in Chapter 7 of van Damme (1987). For any given  $\delta \in (0, 1)$ , define:

$$\alpha: [0, x_1^0] \rightarrow \mathbb{R}$$

such that

$$\alpha(x_1) = (1-\delta)h((1-\delta)x_1) - h(x_1)$$

Then  $\alpha(0) < 0$  and  $\alpha(x_1^0) > 0$ . Since  $\alpha$  is continuous, there is  $x_1^*$  such that  $\alpha(x_1^*) = 0$ .  $(x_1^*, h((1-\delta)x_1^*))$  has the desired property.

Remark: It is pointed out in Chapter 7 of van Damme (1987) that when  $\delta$  goes to 0,  $(x_1^*, x_2^*)$  approaches to a Nash solution. If the

limit point is not the intersection of some  $g_k$  and  $g_{k'}$ , then it is an extended Nash solution.

### Bibliography

- Abraham, R. & J. Robbin, 1967, *Transversal Mappings and Flows* (W. A. Benjamin, New York).
- Damme, E. van, 1987, *Stability and Perfection of Nash Equilibria* (Springer, Berlin).
- Hirsch, M. W., 1976, *Differential Topology* (Springer, New York).
- Intriligator, M., 1971, *Mathematical Optimization and Economic Theory* (Prentice Hall, Englewood Cliffs, N. J.).
- Nash, J. F., 1950, "The Bargaining Problem", *Econometrica* 18, 155-162.
- Nash, J. F., 1953, "Two-Person Cooperative Games", *Econometrica* 21, 128-140.
- Rubinstein, A., 1982, "Perfect Equilibrium in a Bargaining Model", *Econometrica* 50, 97-109.
- Smale, S., 1974a, "Global Analysis IV", *Journal of Mathematical Economics* 1, 119-127.
- Smale, S., 1974b, "Global Analysis V", *Journal of Mathematical Economics* 1, 213-221.
- Suzuki, T., 1987, *General Equilibrium When Some Firms Follow The Full Cost Principle* (Ph. D. Thesis, University of New South Wales).



## ナッシュ交渉モデルの補完に関して：第1部

### 〈要 約〉

鈴木 時 男

本稿においては交渉の数理モデルにおける均衡概念をめぐる問題が提出され、Nash (1953) そしてその拡張としての Rubinstein (1982) が考察の対象となっている。

第1節では利得集合が Smale (1974a) に基づいていくつかの条件を満たす関数で定義される。そしてこのような関数全体の開かつ稠密な部分集合のすべての要素の下で一般化されたナッシュ解が有限個となる事が示される。この結果は Abraham & Robbin (1967) における無限次元の transversality theorem に基礎を置いている。

第2節では Rubinstein model の概要が示され、subgame perfect な解の存在が第1節での一般的な利得集合を用いて証明される。より詳しい議論は稿を改めて行なう予定である。