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THE FINITENESS AND STABILITY OF BARGAINING EQUILIBRIA

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The purpose of this paper is to investigate under what conditions a reasonable number of elements in a parameter space yield a finite number of equilibria which correspond continuously to the change of the parameters. The finiteness of the equilibria will be useful in comparative statics, and the continuous correspondence gives a structural stability to the model.

These problems are studied, using a simple n-person noncooperative game as the basic setting.

This discussion is based on results in Smale (1974a and b), van Damme (1983), and Suzuki (1987).

1. Basic Settings

A game is described by payoff functions u_i: $R^n \to R$, and pure strategy vectors $b_i \in R^{mi}$, $i = 1 \sim n$. n is the number of players, and mi is that of pure strategies for player i. Both are some given numbers. u_i is assumed to be C^1, and u_i and b_i are allowed to change for all i. The whole collection of u_i is defined by $C^1(R^n, R)$, i.e. the space of C^1 maps from R^n to R. $C^1(R^n, R)$ is endowed with C^1 Whitney topology. That is, a neighborhood Nh of the origin is defined by a continuous map h: $R^n \to$ R+, where R+ is the set of strictly positive numbers, such that:

$f \in C^1(R^n, R)$ belongs to Nh
if and only if
$| f(x) | < h(x)$, all $x \in R^n$, and
$\| Df(x) \| < h(x)$, all $x \in R^n$.

As discussed in Chapter 2 of Hirsch (1976), $C^1(R^n, R)$ with the Whitney topology satisfies the Baire property, i.e. any countable intersection of open and dense subsets is dense. If R^n is restricted to a compact set K, then $C^1(K, R)$ with this topology becomes a Banach space.

On the other hand, R^{mi} is defined as the whole collection of pure strategy vectors for player i.

The parameter space of the model is:

$$[C^1(R^n, R)]^n \times \prod_{i=1}^{n} R^{mi}$$

For simplicity, let $A = [C^1(R^n, R)]^n$, $B_i = R^{mi}$, and the Cartesian product of B_i be B.

For any $b_i \in B_i$, a mixed strategy $s_i \in S_i$ is a probability distribution on b_i, where:

$$S_i = \{s_i \in (\overline{R}+)^{mi} : s_i = (s_{i,1}, \sim, s_{i,mi}), \sum_{j=1}^{mi} s_{i,j} = 1\}$$

$\overline{R}+$ is the set of nonnegative real numbers. It should be noted that the nonnegative unit simplex S_i can be considered as a compact smooth manifold (without boundary) of dimension $mi-1$. The tangent space of S_i at any s_i is identified with R^{mi-1}.

Given a map u_i and letting $b_i \in B_i$ and $s_i \in S_i$ be any given element of the domain. Take any component k_j of b_i, and let $s_{i,kj}$ be the corresponding probability. Then the expected payoff function Pu_i is defined by:

$$Pu_i : B \times S \to R$$

such that

$$Pu_i(b.\ s) = \Sigma \left(\prod_{j=1}^{n} s_{i,kj}\right) u_i(k)$$

S is the Cartesian product of S_i, and the summation is over all possible

$k = (k_1, \sim, k_n)$. Pu_i is C^1 by construction. An extended bargaining equilibrium is now defined as an analog of the extended price equilibrium in Smale (1974a).

2. Results with Fixed u

Definition 1

Given $u = (u_1, \sim, u_n)$, let $\sigma Pu_i/\sigma s_i$ be the partial derivative of Pu_i with respect to s_i.

Define:

$$\Psi u : B \times S \to \prod_{i=1}^{n} R^{mi-1}$$

such that

$$\Psi u\,(b, s) = [\sigma Pu_1/\sigma s_1\,(b, s), \sim, \sigma Pu_n/\sigma s_n\,(b, s)]$$

For each $(b, s) \epsilon \Psi u^{-1}(0)$, s is said to be an extended bargaining equilibrium under (u, b). The whole collection of such s is denoted by Eex (u, b).

Next, the regular value theorem, discussed in Chapter 1 of Hirsch (1976), gives the following lemma.

Lemma 1

Suppose that the C^1 map Ψu has 0 as a regular value. That is, $\Psi u^{-1}(0)$ is either empty, or for any $(b, s) \epsilon \Psi u^{-1}(0)$, $D\Psi u\,(b, s)$ is surjective. If this inverse image is nonempty, then it is a C^1 submanifold of $B \times S$, and $\dim \Psi u^{-1}(0) = \dim B$.

Remark:

Similar results are found in Chapter 2 of van Damme (1983). Concerning the existence of an equilibrium, see Chapter 10 of Harsanyi (1977).

Definition 2

Suppose that Eex (u^0, b^0) is a finite set with k elements. Then this set is said to be stable in A × B if there are a neighborhood N of (u^0, b^0) and k continuous functions $\alpha_i : N \to S$, $i = 1 \sim k$, such that for any $(u, b) \in N$, $\alpha_i(u, b) \in$ Eex (u, b), and $\alpha_i(u, b) \neq \alpha_j(u, b)$ if $i \neq j$.

When u is fixed, the stability in B is defined in the same way.

Theorem 1

Suppose that $u \in A$ satisfies the condition in Lemma 1. Fix u. Then there is an open and dense subset θ of B such that for any $b \in \theta$, Eex (u, b) is finite and stable in B.

Proof

Let Π: B × S → B be the projection.
$\Pi | \Psi u^{-1}(0)$ is a C^1 map, and $\dim \Psi u^{-1}(0) - \dim B = 0$. Therefore, the Sard theorem states that the set θ of regular values is open and dense. The inverse function theorem is also applicable, and the restriction is a proper map.

So for any $b \in \theta$, Eex (u, b) must be a finite set with k elements. There are also k C^1 diffeomorphisms between U_i and V_i, where V_i is a neighborhood of b, U_i a neighborhood of some $s \in$ Eex (u, b), $i = 1 \sim k$, and U_i and U_j are disjoint if $i \neq j$.

On the other hand, each V_i can be a subset of a compact set K. $M = (\Pi | \Psi u^{-1}(0))^{-1}(K)$ is compact. Let V be the intersection of V_i, U the union of U_i, $i = 1 \sim k$. The following set difference gives a neighborhood with the desired property:

$$V \setminus (\Pi | \Psi u^{-1}(0))\ (M \setminus U)$$

Remark:

Like Smale (1974b), the discussion using the coordinate space of S_i, $i = 1 \sim n$, will give similar results.

3. Results with Variable u

The discussion here utilizes the results in Chapter 4 of Abraham and Robbin (1967) and Chapter 4 of Suzuki (1987).

Let A, X, Y be C^r manifolds with finite or infinite dimension, $r \geq 0$. A C^r manifold with infinite dimension means that the manifold is a C^r Banach manifold.

Consider any map $\rho: A \to C^r(X, Y)$. This map is called a C^r representation if the following evaluation map is C^r.

$$ev\rho: A \times X \to Y$$

such that

$$ev\rho\,(a, x) = \rho a\,(x)$$

Now let A, X, Y be C^1 manifolds with finite or infinite dimension. In addition, let W be a C^1 submanifold of Y, K a compact subset of X, and $\rho: A \to C^1(X, Y)$ a C^1 representation. Then A_{KW} is an open subset of A, where:

$$A_{KW} = \{a \in A : \rho a \text{ is transversal to W at every } x \in K\}$$

W can be a single element $y \in Y$, i.e. a zero dimensional submanifold. In this case, the transversality is equivalent to the condition that ρa has y as a regular value.

Next, a density theorem is also available.

Define:

$$A_W = \{a \in A : \rho a \text{ is transversal to W}\}$$

Suppose that:

1) X has finite dimension n, and W has finite codimension q in Y.
2) A and X are second countable.

3) $r > \max(0, n-q)$.

4) The evaluation map $ev\rho$ is transversal to W.

Then A_W is residual, i.e. A_W is the countable intersection of open and dense subsets of A. From the Baire property, A_W is dense in A.

Lemma 2

Let Y be the set of $(u, b) \in A \times B$ such that the map $\Psi u,b$ has 0 as a regular value, where $\Psi u,b(s) = \Psi u(b, s)$ for all $s \in S$. The parameter space $A \times B$ and the map Ψu were defined in the previous discussion.

Then Y is open and dense in $A \times B$.

Proof

First, consider the openness. Take any $(u^0, b^0) \in Y$, where $u^0 = (u_1{}^0, \sim, u_n{}^0)$ and $b^0 = (b_1{}^0, \sim, b_n{}^0)$.

Let K_i be any compact set containing b_i^0. It is also possible to choose K_i which is a smooth manifold of dimension $mi-1$. Given K_i, the domain of u_i, i.e. R^n, can also be restricted to some compact smooth manifold K' of dimension n, for all i. Let A^* be A with the restricted domain, and K the Cartesian product of K_i, $i = 1 \sim n$.

Then the following map becomes a C^1 representation:

$$\Psi^* : A^* \times K \to C^1(S, \prod_{i=1}^{n} R^{mi-1})$$

such that

$$\Psi^* u^*,b\,(s) = \Psi u^*(b, s), \text{ all } s \in S$$

Since S is compact, the openness theorem states that there is a neighborhood N^* of (u^{*o}, b^o) in $A^* \times K$ such that for any $(u^*, b) \in N^*$, $\Psi^* u^*, b$ has 0 as a regular value.

Now the following map f is easily shown to be continuous, where the domain and the codomain are endowed with the C^1 Whitney topology:

$$f : C^1(R^n, R) \to C^1(K', R)$$

such that

$$f: u_i \rightarrow u_i \mid K'$$

Therefore, there is a neighborhood N of (u^0, b^0) in $A \times B$ with the desired property, which proves the openness of Y.

Next, consider the denseness of Y. Let $b \in B$ be any element.

Define:

$$A(b) = \{u \in A : \Psi u,b \text{ has } 0 \text{ as a regular value}\}$$

Then it follows from the previous discussion that A (b) is open in A. The density theorem is also applicable to A(b) with the restricted domain. Since $f: u_i \rightarrow u_i \mid K'$ is also an open map, A (b) is concluded to be dense in A.

Now let B′ be any countable and dense subset of B. For example, let D be the set of rational numbers, and B′ the Cartesian product of D^{mi}, $i = 1 \sim n$. Then from the Baire property, the countable intersection of A (b), $b \in B'$, is dense in A.

In addition, the Cartesian product of the intersection and B′ must be a subset of Y. This gives the denseness of Y and completes the proof.

By construction, Eex (u, b), $(u, b) \in Y$, is a finite set. The stability is given by the following lemma.

Lemma 3

There is an open and dense subset O of Y such that for every $(u, b) \in O$, Eex (u, b) is stable in Y.

Proof

The statement is based on Smale's version of the Sard theorem, discussed in Smale (1965). Since Y is open in $A \times B$, Y is expressed as a union of the sets $Y_{1,k} \times Y_{2,k}$, where $Y_{1,k}$ is open in A, and $Y_{2,k}$ is open in B. It is also possible to assume that for each k, $Y_{2,k}$ is bounded.

Since the stability is a local question, the discussion will be restricted to $Y_{1,k} \times Y_{2,k}$. For simplicity, the subscript k will be omitted.

Since Y_2 is bounded, the domain of u_i, $i = 1 \sim n$, can be restricted to a compact smooth manifold of dimension n. Let Y_1^* be Y_1 with the restricted domain.

Define:

$$\Psi^{**} : Y_1^* \times Y_2 \times S \rightarrow \prod_{i=1}^{n} R^{mi-1}$$

such that

$$\Psi^{**}(u^*, b, s) = \Psi u^*(b, s)$$

Clearly, this C^1 map has 0 as a regular value. Then from the infinite dimensional version of the regular value theorem, discussed in Chapter 4 of Abraham and Robbin (1967), $(\Psi^{**})^{-1}(0)$ must be a C^1 submanifold of $Y_1^* \times Y_2 \times S$, and codim $(\Psi^{**})^{-1}(0)$ is equal to $\sum_{i=1}^{n}(mi-1)$.

Consider the restriction $\Pi \mid (\Psi^{**})^{-1}(0)$ of the projection $\Pi : Y_1^* \times Y_2 \times S \rightarrow Y_1^* \times Y_2$, and take any (u^*, b, s) of $(\Psi^{**})^{-1}(0)$.

The derivative $D\Pi(u^*, b, s)$ is considered to be the following composition:

$$p \cdot i : [(\Psi^{**})^{-1}(0)]_{(u^*,b,s)} \rightarrow [C^1(K', R)]^n \times \prod_{i=1}^{n} R^{mi} \times \prod_{i=1}^{n} R^{mi-1} \rightarrow [C^1(K', R)]^n \times \prod_{i=1}^{n} R^{mi}$$

$[(\Psi^{**})^{-1}(0)]_{(u^*,b,s)}$ is the tangent space of the C^1 submanifold at (u^*, b, s), i is the inclusion map, and p is the projection p: $(\tilde{u}, \tilde{b}, \tilde{s}) \rightarrow (\tilde{u}, \tilde{b})$.

Ker i = $\{0\}$, the range of i is closed in the codomain, and the dimension of the following quotient space is $\sum_{i=1}^{n}(mi-1)$.

$$[C^1(K', R)]^n \times \prod_{i=1}^{n} R^{mi} \times \prod_{i=1}^{n} R^{mi-1}/\text{Im } i$$

i is thus a Fredholm operator whose index is $-\sum_{i=1}^{n}(mi-1)$. On the other hand, Ker p is $\{0\} \times \prod_{i=1}^{n} R^{mi-1}$, the range of p is closed in the codomain, and the dimension of the following quotient space is zero.

$$[C^1(K', R)]^n \times \prod_{i=1}^{n} R^{mi} / \text{Im } p$$

p is thus a Fredholm operator whose index is $\sum_{i=1}^{n}(mi-1)$. Then from the discussion in Chapter 2 of Istratescu (1981), the composition T • S of two Fredholm operators T, S is a Fredholm operator. The index of T • S is equal to the summation of the index of T and that of S.

Therefore, the index of $D\sqcap(u^*, b, s)$ is zero at any (u^*, b, s) of $(\Psi^{**})^{-1}(0)$. It can be concluded that the restriction $\sqcap | (\Psi^{**})^{-1}(0)$ is a C^1 Fredholm map. Using Smale's version of the Sard theorem, the following result is obtained:

> The set of regular values of $\sqcap | (\Psi^{**})^{-1}(0)$ is open and dense in $Y_1^* \times Y_2$.

$[(\Psi^{**})^{-1}(0)]_{(u^*,b,s)}$ and $[C^1(K', R)]^n \times \prod_{i=1}^{n} R^{mi}$ are subspaces of

$$[C^1(K', R)]^n \times \prod_{i=1}^{n} R^{mi} \times \prod_{i=1}^{n} R^{mi-1}.$$

Since these subspaces have the same codimension $\sum_{i=1}^{n} R^{mi-1}$, the infinite dimensional version of the inverse function theorem is applicable. It is also easily seen that $\sqcap | (\Psi^{**})^{-1}(0)$ is a proper map.

By using the open and continuous map f: $u_i \to u_i | K$ again, the following result is obtained:

> There is an open and dense subset Z of $Y_1 \times Y_2$ such that for any $(u, b) \in Z$, Eex (u, b) is stable in $Y_1 \times Y_2$.

This completes the proof.

Therefore, the following theorem is obtained as a straightforward consequence of these results.

Theorem 2

There is an open and dense subset θ of A × B such that for any (u, b) ϵ θ, Eex (u, b) is stable in A × B.

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ゲーム解の有限性と安定性
〈要　　約〉

鈴　木　時　男

本稿においては数理経済学における均衡概念をめぐる問題が提出され，基本的なｎ人非協力ゲームをモデルとして，与えられた条件とその下での均衡解との関係が論じられる。特にどのような仮定の下ならば，十分に多くの初期条件の組み合わせにおいて均衡解の集合が有限個の要素を持ち，かつその組み合わせの変化に連続的に対応するかが議論される。パラメータとして個人ｉ，ｉ＝１～ｎ，の利得関数と純粋戦略が設定され，その集合はそれぞれR^nからＲへのC^1関数の全体とmi 次元のユークリッド空間として表現される。

均衡はナッシュ解を一般化したものが用いられ，この均衡を与える関数がゼロベクトルを正則値(regular value)として持つことが条件として提出される。

第１に，この条件下で均衡解集合の有限性が示される。

第２に，パラメータ集合の開かつ稠密な部分集合が存在し，そのすべての要素がこの正則値の条件を満たす関数を構成することが示される。

第３に，これらの要素に関して均衡解の連続的対応が証明される。

以上の議論は Smale (1974a and b), van Damme (1983), そして Suzuki (1987) の成果に基礎を置いている。