『社会科学ジャーナル』29(2) 〔1991〕 pp.21-36
The Journal of Social Science 29(2) 〔1991〕 ISSN 0454-2134

NATURE'S WAYS ARE INVARIANT

Ryuzo Sato*

1. Hsün Tzu (Ca. 305–235 B. C.), an ancient Chinese sage, once stated that "Nature's ways are invariant." Nature's ways, including human behavior, follow certain regular and systematic patterns amid irregular and unpredictable forces. Economics is the study of regularities of human economic behavior and the identification of economic rules and economic laws. Economic laws are the description of "invariant" relationships among relevant variables.

The Lie group theory is a modern tool in economics designed to identify economic invariances and economic laws arising from optimal economic behavior. The theory has successfully been applied to uncover both apparent and hidden economic invariances in such areas as technical change, economic conservation laws, and dynamic symmetries, duality, and economic index numbers, to cite a few examples.

2. To demonstrate what is meant by a Lie group and to say why Lie groups are relevant, let us consider a typical estimation problem of the underlying production function and technical change. Assume that technical progress in the production process is a priori known to have the simple "neutral" form,

$$T_t: \overline{K}=e^{\alpha t}K, \quad \overline{L}=e^{\alpha t}L,$$

* C. V. Starr Professor of Economics, New York University

where K is the capital, L the labor, α the rate of technical progress ($\alpha \geq 0$), $\bar{K}$ the "effective" capital, $\bar{L}$ the "effective" labor, and t the index of technical progress. The equations for $\bar{K}$ and $\bar{L}$, which may be called the technical progress functions for capital and labor, constitute a *one-parameter Lie group of continuous transformations* (Lie [1891]). Let the parameter of technical progress t change from t_0 to t_1. Then $\bar{K}$ and $\bar{L}$ change from

$$T_{t0}: \bar{K}_0 = e^{\alpha t0}K,\ \bar{L}_0 = e^{\alpha t0}L, \text{ to } T_{t1}: \bar{K}_1 = e^{\alpha t1}K,\ \bar{L}_1 = e^{\alpha t1}L.$$

The technical progress functions constitute a Lie group for the following reasons:

(i) (*Composition*) The result of the successive transformations of T_0 and T_1 is the same as that of the single transformation.

$$T_{t2}: \bar{K}_2 = \exp(\alpha(t_0 + t_1))K,\ \bar{L}_2 = \exp(\alpha(t_0 + t_1)L.$$

(ii) (*Identity*) When there is no technical change, i.e. $t=0$, then $\bar{K}=K$ and $\bar{L}=L$.

(iii) (*Inverse*) The inverse functions of T_t are also a member of T, when t is replaced by $-t$,

$$T_t^{-1} = T_{-t}: K = e^{-\alpha t}\bar{K},\ L = e^{-\alpha t}\bar{L}.$$

From the aggregate of the transformation included in the family T_t, where t varies continuously over a given range, any particular transformation of the family is obtained by assigning a particular value to t. Any successive transformations (including identity and inverse transformations) of the family are equivalent to a single transformation of the family. These are the basic properties of a Lie group.

Now assume that the estimation equation is derived from the market observation on the marginal rate of substitution between capital K and labor L by

$$P_K/P_L = Y_K/Y_L = f(K/L, t),$$

where P_K is the price of capital, P_L the price of labor, Y the output, $Y_K = \partial Y/\partial K$ the marginal product of K, and $Y_L = \partial Y/\partial L$ the marginal product of L. If $\bar{K}$ and $\bar{L}$ are related with K and L by the technical progress functions T_t given in the foregoing and if T_t is the only source of technical progress of the system, then it is seen immediately that the estimated marginal rate of substitution f should not contain t, because f coincides with the quantity known as the *invariant* of the group, i.e.,

$$f(K/L,t)=f(\bar{K}/\bar{L})=f(e^{at}K/e^{at}L)=f(K/L).$$

This means that the efficiency increase of capital and labor cannot be estimated from the observed behavior of the marginal rate of substitution. Furthermore, from the behavior of f, it is "impossible" to identify any "economies of scale" even if they exist. This is because the underlying production function is a member of the so-called *invariant family* of curves generated by this group.

In general, given a Lie type of technical progress T_t, one can always derive a family of production funtions invariant under T_t (holothetic technology). Conversely, given the observable marginal rate of substitution in the form of a differential equation

$$M(K,L)dK+N(K,L)dL=0 \text{ or } -\frac{dL}{dK}=\frac{P_K}{P_L}=\frac{M(K,L)}{N(K,L)}=\frac{Y_K}{Y_L},$$

there exists a one-parameter Lie group of transformations (Lie type of technical change) which leaves the underlying production function invariant. If we know beforehand how this type of technical change acts on capital and labor, we can use this knowledge to find the underlying production function and to study its properties. This is an important reason why we may want to study the application of Lie groups.

Another example may be taken from the area of dynamic optimization. Jürgen Moser in his address at the 1978 annual meeting of the National Academy of Sciences states:

> I should like to present a glimpse into a new area of mathematics-one in which remarkable development has taken place over the last fifteen years ... it is concerned with symmetries in dynamic systems that are not apparent at first and are only revealed through deep analysis. Such terms as ,..., integrable dynamic systems, and conservation laws fall into this area (Moser [1979]).

Paul A. Samuelson [1970] was the first to formally introduce "conservation laws" in theoretical economics. Samuelson has in effect shown that the maximized value of the Hamiltonian in a simple optimal growth model of the von Neumann type is constant and *invariant* under a Lie translation group of the time variable — the "conservation law of the (aggregate) capital-output ratio." The translation group is the simplest type of Lie groups. Suppose one can find another more general Lie group (technical change) under which the dynamic system is *invariant*, then by Noether's [1918] theorem, one can derive another fundamental economic law of conservation. This is basically the study of the relationship between the (Lie) group invariances (often referred to as *dynamic symmetries*) of a system and its integrals, or "conservation laws." Economists have not yet been fully exposed to this aspect of dynamic analysis: they have been too busy devising economic interpretations of only the Euler-Lagrange equations.

The theory of Lie groups was developed by the Norwegian mathematician Sophus Lie in the late 19th century in connection with his work on systems of differential equations. Lie groups arise in the study of solutions of differential equations just as finite groups arise in the study of algebraic equations. The possibility of solving many differential equations which arise in practical problems can often be traced to some geometrical or other symmetry property of the problem. Indeed, Lie groups play a fundamental role in many branches of geometry itself and have contributed significantly to the development of differential geometry of symmetric spaces and to abstract analysis (Chevalley [1946]).

The theory of Lie groups and Lie algebras is an area of mathematics in which we can see a harmony between the methods of classical analysis and modern algebra (e.g., Nono [1968]). The application of the Lie theory is one of the most powerful and most systematic approaches to the theory of invariant behavior. Belinfante and Kolman [1972, p.vii] observe: "Applications of the Theory of Lie Groups ... are many and varied. This is a rapidly growing field through which one can bring to bear many powerful methods of modern mathematics." These fields include: "differential equations and their solvability; quantum mechanics and symmetries; harmonic oscillators; lattice; representation theory; analysis of integral variational problems; Hamiltonian approach conservation laws and equations of motion; etc."

3. I will present in some detail more specific analysis of the issues. We begin with the study of technical progress and production functions. Specifically, we address the question of what the impact of technical progress is upon different economic variables. At the forefront of this topic is the Solow-Stigler controversy: that is, can technical progress effects be independently isolated and identified from scale effects (i.e., growth in the capital-labor ratio)? Solow has argued that the portion of the increase in U.S. per capita output that is not explained by growth in the capital-labor ratio may be attributed to technical progress. In his econometric estimations, he assumes that technical progress is of the Hicks neutral type and that the underlying production function is linearly homogeneous (constant returns to scale). Stigler, on the other hand, has criticized the assumption of linear homogeneity of the production function and has emphasized the necessity of distinguishing between increasing returns to scale and technical progress.

We shall establish conditions under which the effects of technical progress and scale effects are idependently identifiable. We begin with the definition of technical progress functions (ϕ and ψ). When technical progress is introduced into a production process, it changes the way in which factor inputs, capital (K) and labor (L), are combined. These technical progress functions combine the factor inputs through the technical progress parameter t.

$$\bar{K}=\phi(K,L,t),\ \bar{L}=\psi(K,L,t).$$

By using this definition, the new — and fundamental— concept of *holotheticity* is introduced (see Sato [1975]). Specifically, when the entire effect of technical progress can be represented by some monotonic transformation F, then the production function is said to be holothetic under that given type of technical progress. If we impose the condition that the technical progress functions possess the Lie group properties, then we can find a family of production functions under which the total effect of technical progress is completely transformed to apparent scale effects – ***holothetic technology.***

In response to the Solow–Stigler controversy, we are now able to conclude that the effects of a given type of technical progress and scale effects are independently identifiable if and only if the production function is not holothetic under that particular type of technical progress. For example, since the well–known homothetic technology is holothetic under the uniform factor–augmenting type of technical progress, but not under the nonuniform type, scale effects and technical progress effects cannot be isolated in the first case; whereas in the latter situation under the nonuniform type, they are independently identifiable.

In practice, it is convenient to express technical progress functions as infinitesimal changes in the technical progress parameter. On using Lie's own notation, the *infinitesimal generator* is introduced ($U=\Sigma^n_{i=1} \partial\zeta_i(x)/\partial x_i$). Given this infinitesimal transformation, the holothetic technology is simply derived as the family of curves that are invariant under the given group of technical progress transformations. As is well known, the optimal behavior of a cost–minimizing firm can be observed in the marketplace by studying the differential equation for the marginal rate of substitution (*MRS*). Given the Lie "representation" of the *MRS*, it is possible to test if a particular production technology is holothetic under a given Lie type of technical progress by the *compatibility condition* of the infinitesimal generators.

Of the several well–known types of technical progress that have appeared in economic literature over the years, the most frequently encountered is the Hicks neutral type. It is demonstrated that the Lie type of technical progress under its own holothetic production function can always be expressed as a Hicks neutral type of product–augmenting technical progress. Since any production function has at least one type of technical change under which it is holothetic, this would appear to be a reasonable justification for using Hicks neutral technical change in empirical studies. Historically, though, the Hicks neutral type has been used because of its presumed "neutral" effect upon the distributive shares of the factors of production, given linearly homogeneous technology. Specifically, technical progress is Hicks neutral if the *MRS* is invariant under technical change, as long as the capital-labor ratio is invariant.

The concept of "neutrality" is basically the same as the group concept of invariance. Specifically we extend the concept of "neutrality" used in the earlier works to "neutrality in the sense of transformation groups" — "G (group)-neutral" type of technical change. If we now introduce two parameters of technical change (α and β), then technical change may be represented by a family of neoclassical production functions of the form $Y=F(K,L,\alpha,\beta)$. If y is used to represent the output-capital ratio and x the labor-capital ratio, then the production function may be written as $y=f(x,\alpha,\beta)$. Using this framework, we state that technical change expressed in this form is G neutral if this production function is invariant under a Lie transformation group (G) of r essential parameters. Specifically, for a given G, the second-order equations of neutrality can be obtained using the condition that for all $U \in L(G)$ (where $L(G)$ is the Lie algebra of infinitesimal transformations of G), $U''\phi=0$ whenever $\phi(x,y,y_x,y_{xx})=0$. The family of production functions of G-neutral technical change can be obtained by simply solving this equation of neutrality. The approach taken here can not only justify the well-known types of neutral technical change, but also generate more meaningful and more general types of technical progress.

4. The versatility of applications of the Lie theory does not end here. It is demonstrated that we can further apply this theory to the study of duality and self-duality of preferences and technologies. After presenting the necessary and sufficient conditions for the self-duality of preferences as the set-symmetric conditions on the implicit functions of price and quantity vectors, we present a specific method of deriving such implicit functions. We consider the demand functions with r

essential parameters as continuous transformations satisfying the budget constraint. If we place certain fundamental restrictions on these demand functions, we can assume that they satisfy the Lie group properties. It turns out that the self-dual demand functions that in fact satisfy these fundamental restrictions are simply the continuous transformations of the unitary elastic demand functions associated with a Cobb-Douglas preference ordering. In other words, we can make use of the fact that the system of demand functions arising from a Cobb-Douglas utility function with equal exponents can be used as the basis for the identical transformation.

We know from the theory of Lie groups that there are r linearly independent infinitesimal transformations associated with the demand functions. Given these infinitesimal transformations, the self-duality conditions, especially for a separable system, are stated as the invariance conditions of the group. Hence the *invariants* of the group together with the budget constraint constitute a separable system of set-symmetric functions of the self-dual demand system.

Although the basic duality principle of the utility and demand analysis will carry over to production and cost analysis, the self-duality of production and cost functions is usually different. The main reason for this is that here we are *not* comparing the direct and indirect production functions. Using the normalized cost function, we first state the necessary condition for the self-duality of the production and cost function: The production function must be *implicitly holothetic* under the uniform factor-augmenting type of technical progress. The observant reader will note that this implies that the production function must be implicitly homothetic. Hence we must deal with the problem of *implicit self-duality.* It is shown that such an implicitly homothetic production fuction has a cost function of a particular form. If and only if the production function is implicitly homothetic can the cost function be written in the form $C=g(C_1(Y),\ldots,C_{n-1}(Y);p)C_n(Y)$. Using this, we formulate the necessary and sufficient conditions for the implicit self-duality of technologies.

The uniformity of factor demand functions is defined as the similarity of the functional forms of all demand functions. J. B. Clark's treatment of the marginal productivities of capital and labor clearly assumes this special property for the factor input demand functions (Clark uniformity).

One advantage of Lie group is to facilitate the analysis of economic conservation laws. Noether's theorem is the essential tool for this purpose. The theorem states that "if the fundamental integral of a problem in the calculus of variations and optimal control is invariant under the r-parameter (Lie) group of transformations, then there are r conservation laws." The Hamiltonian canonical transformation leaves a dynamic system invariant. By further extending this concept one can derive Noether's invariance identities, which are the system of partial differential equations involving the Lagrangian of the model, its derivatives, and the infinitesimal transformations. These identities can be used in two ways: Given the Lagrangian, an r-parameter group will be generated through the solutions of the partial differential equation system; given the r-parameter group, the corresponding Lagrangians will be integrated again through these solutions.

The invariant variational principle is applied to general neoclassical optimal growth models of the Ramsey type. It is shown that there exist several (local) conservation laws in the neighborhood of the steady state. Noether's theorem applied to a typical problem of welfare maximization over time has an interesting economic interpretation. If a welfare function is dynamically invariant under an r-parameter family of transformations resulting from technical change and/or taste change, then the following expression is constant along any optimal path for the entire period:

measure of welfare		value (effect)
per infinitesimal	+	of technical (taste) change
change of time		for the ith quantity
	+	null term = const.

One of the most interesting aspects of the conservation laws is that the Hamiltonian itself is not constant when the welfare function is discounted with a positive rate. However, in the neighborhood of the equilibrium position, the discounted welfare measured in terms of the "modified" Hamiltonian, which is the sum of the Hamiltonian and the value of technical change, remains unchanged.

Economic conservation laws and turnpikes are closely related. Thus the two additional local laws are the turnpike constants of the system. The weighted difference between investment and capital multiplied by the inverse of the negative turnpike exponent, and the weighted sum of investment and capital multiplied by the inverse of the positive turnpike exponent, are always constant.

Finally, the Samuelson conservation laws in a von Neumann growth model is derived through the application of the Noether's theorem. It is shown that the Samuelson laws are the only laws globally operating for that system, although there are several local conservation laws. Again the turnpike constants are shown to be closely related with the local conservation laws.

The following table (Sato and Maeda [1990]) summarized the existing results on economic conservation laws:

	Lagrangian	*Infinitesimal Transformation*	*Conservation Laws*	*Examples*
Model I	$L=L(x(t)),\dot{x}(t))$	$\tau=1$ $\zeta=0$	H=Wealth Measure of National Income =constant	Original Ramsey Model
	$L=\dot{K}_1+\lambda F(K,\dot{K})$	$\tau=\gamma$=constant $\zeta=\alpha K$ $w=-\alpha\lambda$ $\Phi=\alpha K_1+C$ α,C=constant	λY=constant λW=constant i.e., $\frac{Y}{W}=\frac{\text{output}}{\text{wealth}}$ =constant	von Neumann-Samuelson Model
Model II	$e^{-\rho t}L(x(t),\dot{x}(t))$ $\rho>0$	$\tau=1$ $\zeta=0$ $\Phi\neq 0$ $\frac{d\Phi}{dt}=-\rho e^{-\rho t}L$	Income-Wealth Conservation Law =Discounted Income+ρ	Neoclassical Growth Model (Weitzman)

			×Discounted Stock of Consumption $=\rho\times$Max Discounted Stock of Consumption =constant	Neoclassical Theory of Investment Endogenous Theory of Technical Change
Model III	$e^{-\rho(t)}L(x,\dot{x})$	$\tau=\frac{1}{\rho'(t)}$ $\zeta=0$	Income$=\rho'(t)$ × "generalized" wealth	Variable Discount Rate (Samuelson, Sato)
Model IV	$e^{-\rho t}L(x,\dot{x},t)$ $=e^{-\rho t}L(x,e^{\rho t}\dot{x})$	$\tau=e^{\rho t}$ $\zeta=0$ =constant	Current Hamiltonian	"Factor-Aug. Technical Change on k" (Sato)
	General Case $e^{-\rho t}L(x,\dot{x},t)$	$\tau=e^{\rho t}$ $\zeta=b(t)\times$ $\exp\int_0^t\left(-\frac{L_x}{L_{\dot{x}}}ds\right)$	Income + "Value of Taste (Technical) Change" $=\rho\times$wealth	General Technicl and Taste Change (Sato, Nôno and Mimura)
		$\tau=0$ $\zeta\neq0$	Modified Supply Price of Investment	*same as above*
Model V	$e^{-\rho t}[Q]$ Q=Quadratic in x and $\dot{x}$	$\tau=\delta$=constant $\zeta=\frac{\rho\delta}{2}\zeta$ =constant	Modified Hamiltonian: Income$+\rho\times$ value of capital	Near the Steady-State (Sato)
Model VI	$\Sigma\lambda^{-t}L_t$ L_t=Quadratic in $x(t)=q(t)-q^*$ and $x(t+1)-x(t)=v(t)$ $\lambda=1+\delta$		Modified Hamiltonian (1)Discrete Model Modification and (2)Discount Factor Modification $\lambda^{-t}\Big[(-v'\frac{\partial L}{\partial v}-L)$ $+1/2v'\frac{\partial L}{\partial q}$ $+\frac{(\lambda-1)}{2}q'\frac{\partial L}{\partial q}\Big]$ =constant	Discrete Dynamic System (Local Approximation) (Sato-Maeda)

Concluding Remarks

Modern economics employs many modern tools: Optimal control theory, static and dynamic game theory, Lie groups etc. I have briefly summarized some of the areas where invariance properties play a key role in economic analysis. Index number problems, which have attracted the attention of many competent economists, have also been subjected to the rigorous application of the Lie group theory. Public finance theory can also benefit from modern applications of newer techniques. "Neutrality" of taxation is the right subject which can be reformulated by the group theory.

References

Baumol, W. [1977], *Economic Theory and Operations Analysis*, 4th ed., Prentice-Hall, Englewood Cliffs, New Jersey.

Belinfante, J. G. F. and Kolman, B. [1972], *A Survey of Lie Groups and Lie Algebras with Applications and Computational Methods*, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

Chevalley, C. [1946], *Theory of Lie Groups*, Princeton Univ. Press, Princeton, New Jersey.

Fuss, M. and MacFadden, D. (eds.) [1978], *Production Economics: A Dual Approach to Theory and Applications*, Vols. I and II, Contribution to Economic Analysis, No. 110, North-Holland Publ., Amsterdam.

Hoffman, W. C. [1968], "The Neuron as a Lie Group Germ and a Lie Product", *Quarterly of Applied Mathematics*, 25, 423-440.

Lie, S. [1891], *Vorlesugen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen*, bearbeitet und herausgegeben von Dr. Georg Scheffers, Leipzig (reprinted by Chelsea, New York, 1967).

Moser, J. [1979], "Hidden Symmetries in Dynamical Systems", *American Scientists*, 67, 689-695.

Noether, E. [1918], "Invariante Variationsprobleme", *Nachrichten von der Akademie der Wissenschaften in Göttingen, Mathematisch-Physikalische Klasse II*, 235-257 (translated by M. A. Travel under the title "Invariant Variational Problems", *Transport Theory and Statistical Physics*. 1, 186-207, 1971).

Nono, T. [1968], "On the Symmetry Groups of Simple Materials: Application of the theory of Lie groups," *Journal of Mathematical Analysis and Applications*, 24, 110-135.

Ramsey, F. [1928], "A Mathematical Theory of Saving", *Economic Journal,* 38, 543–559.

Samuelson, P. A. [1970], "Law of Conservation of the Capital-Output Ratio", Proceedings of the National Academy of Sciences, *Applied Mathematical Science,* 67, 1477–1479.

Sato, R. [1975], "The Impact of Technical Change on the Holotheticity of Production Functions", presented at the World Congress of the Econometric Society in Toronto, Canada; *Review of Economic Studies,* 47(1980), 767–776.

Sato, R. [1981], *Theory of Technical Change and Economic Invariance: Application of Lie Groups,* Academic Press, New York.

Sato, R. and Maeda, S. [1990], "Conservation Laws in Continuous and Discrete Models" in *Conservation Laws and Symmetry,* ed. by Sato, R. and Ramachandran, R. Kluwer Academic, Boston.

天 行 有 常

〈要　　約〉

佐 藤 隆 三

中国の哲人，荀子が述べた「天行有常」という言葉が意味するのは，不規則であるように観察される自然の動きの背後に，ある系統的な不変のパターンが存在するということである。近年，経済学においても，様々な変数の中に不変の経済法則を見出そうとする動向がみられる。次にあげる諸要因はその動向の源泉となっている。

1. 経済発展を「新たな状態への移行」と解釈することに対する反省が生じた。
2. 自然科学分野で不変性 (invariance) を追及し始めたことが経済学に影響した。
3. 情報の対称性の概念が議論されるようになった。
4. 対称性の概念が不変性と一致する。

諸要因を総合し理論化された経済法則を抽出することは，経済的最適状態を求めることに繋がる。

本稿では，経済法則の分析手段としてのリー (Lie) 群理論について論じる。元来，リー群理論は微分方程式の解を求める研究から発見された数学的理論であるが，不変性の分析に最適なツールとして経済学にも幅広く応用されている。そこで，まずリー群の技術進歩と生産関数の関係，ソローースティグラー (Solow-Stigler) 論争，無限小変換記号について解説する。次に中立性が基本的に不変性と同じ概念であることを示す。更に，消費・生産問題上の諸関数が自己双対性の必要十分条件を満足するならば，陰関

数を導出できることを詳説する。最後に，リー群理論は経済保存則を求める上で重要な役割を果たすことを強調する。