

Multiple Equilibria and Dynamics in a General Disequilibrium Model

Taisei Kaizoji

Abstract

The purpose of this paper is to demonstrate (1) the existence of Walrasian equilibria, (2) necessary and sufficient conditions for dynamic stability of a Walrasian equilibrium, and (3) a sufficient condition for *discrete-time disequilibrium dynamic process* to lead to chaos in a general disequilibrium model which has stochastic rationing and price flexibility. The results suggest, that the propensities of disequilibrium dynamics depend upon the speeds of adjustment of prices and quantities, and upon the number of Walrasian equilibria.

Key words : A discrete-time disequilibrium dynamics; Multiple equilibria; Stochastic rationing; Chaos.

1. Introduction

Over the past few decades a considerable number of studies have been made on the non-Walrasian economy. Major contributions to economic theory have been devoted to modelling economic activity, when prices are rigid [Clower (1965), Barro and Grossman (1971), Benassy (1975) and Dreze (1975)]. Benassy (1975) and Dreze (1975) proved the existence of the non-Walrasian equilibria in the general quantity constraint model. It is well known by now that the effective demands derived from deterministic constraints (the Clower-Benassy or Dreze effective demand) may be not consistent with the maximization of utility. This difficulty led researchers to the idea of *stochastic rationing schemes*, suggested first in Benassy (1977). As shown by Gale (1979), Green (1980), and Svensson (1980, 1981), the effective demands with stochastic rationing are consistent with the maximization of expected utility. The macroeconomic implications of the idea

were investigated by Honkapohja and Ito (1985). Furthermore, Matsumoto (1993,1994) studied the complex dynamics in a simple disequilibrium macroeconomic model. He considered a fixed-price economy in which demand is not compatible with supply and in which an individual's optimal behavior is subject to stochastic rationing. In such environments, he demonstrated that the adjustment process is inherently nonlinear and generates complex dynamics involving chaos through the interaction of the individuals' behavior in different markets. However the effective excess demands evidently influence the evolution of prices over time. Therefore, the movement of prices and quantity from one period to the next, namely, *disequilibrium dynamics*¹ should be an important problem to be solved in the disequilibrium models.

This paper examines the dynamics of a *discrete-time disequilibrium dynamics* in a general disequilibrium model with the stochastic rationing and prices flexibility and demonstrates the existence of Walrasian equilibria, and necessary and sufficient conditions for dynamic stability of a Walrasian equilibrium, and sufficient conditions for the disequilibrium dynamics to lead to complex dynamics by applying the Hata theorem [Hata (1982)]².

The fundamental structure of this model is similar to that of Honkapohja and Ito. In the model stochastic effective demands are determined by maximization of expected utility or expected profit subject to stochastic rationing. Since individuals cannot realize their desired transactions in such environments, the actual transactions are generally different from the expected transactions. These disequilibria cause macroeconomic dynamics in the markets. We consider an economy in which there are many goods and many factors, and in which there are the multiple equilibria. In such an environment, we demonstrate that if the adjustment speeds of prices and quantity are sufficiently fast, then a disequilibrium dynamics is chaotic in the sense of Li-Yorke.

In section 2 and 3, the structure of a general disequilibrium model and the stochastic rationing mechanism are specified. Conditions of multiple equilibria and chaos, respectively, are demonstrated in section 4 and 5. A few concluding remarks are given in section 6.

2. Model

Consider an economy in which there is a finite number of states, $s = 1, 2, \dots, S$, that are realized at the end of a period, and in which there are k goods (outputs) and $(z - k)$ factors (inputs). Letting p_n be the price of the n^{th} good, $n = 1, 2, \dots, k$, and p_m the price of the m^{th} factor, $m = k + 1, k + 2, \dots, z$, these prices are summarized by the vector :

$$p = (p_1, p_2, \dots, p_k, p_{k+1}, \dots, p_z). \quad (1)$$

We define the *market signals* as the aggregate excess demand in the markets that agents expect at the beginning of a period. The *market signals* in the n^{th} good market and the m^{th} factor market, respectively, are denoted by ξ_n , and ξ_m . Letting ξ be the vector of the disequilibrium signals of good markets and factor markets :

$$\xi = (\xi_1, \xi_2, \dots, \xi_k, \xi_{k+1}, \dots, \xi_z). \quad (2)$$

that

Let us consider the following situation :

At the beginning of the period t , all households and firms in a competitive market are given the information of prices. No agent can know the state s that is realized at the end of a period. It means that the agents cannot know the actual demand and the actual supply certainly. Accordingly an agent forecasts the market conditions and has expectations with respect to the market signals that are defined as the aggregate excess demand in the market at the beginning of the period t . Based on the expectations, an agent forms his subjective probability distribution of proportion that he is allowed to trade at the decision-making point in time. The agents' offer that is *effective demands* which take stochastic constraints into account are calculated as a result of expected utility (or profit) maximization. Then some proportion of each offer is realized, where the rationing proportion is stochastic.

2.1 Households

In an economy there are H households, $h = 1, 2, \dots, H$, (each of which owns certain factors) and F firms, $f = 1, 2, \dots, F$, (each of which purchases inputs from factor markets to produce outputs). Each household can own the shares of firms and thereby receive a

portion of their profits. The total income from sales of factors and ownership of firms is used to be purchased by household.

We let :

c_n^h is the h^{th} household's effective demand for n^{th} good, $n = 1, 2, \dots, k$,

r_m^h is the effective supply of the m^{th} factor sold by the h^{th} household, $m = k+1, k+2, \dots, z$,

π_s is the probability-beliefs of an agent in the state s , $s = 1, 2, \dots, S$, $\pi_s = \pi_s(\xi) > 0$, and $\sum_{s=1}^S \pi_s(\xi) = 1$,

$\varrho_{n,s}^h$ is the rationing proportions to the h^{th} household in the n^{th} good market and in the state s ,

$\sigma_{m,s}^h$ is the rationing proportions to the h^{th} household in the m^{th} factor market and in the state s ,

$\varrho_s^h c^h$ is the row vector of the actual trade in the n^{th} good market perceived by h^{th} household in the state s is $\varrho_{n,s}^h c_n^h$,

$\sigma_s^h r^h$ is the row vector of the actual trade in the m^{th} factor market perceived by h^{th} household in the state s is $\sigma_{m,s}^h r_m^h$.

The utility of h^{th} household, which depends on both goods consumed and factors supplied, is :

$$U^h = U^h(\varrho_s^h c^h, \sigma_s^h r^h). \quad (3)$$

We assume *voluntary exchange*, that is, the h^{th} household is never forced to buy or sell more than his offers in all the markets and in all the states:

$$0 \leq \varrho_{n,s}^h \leq 1, \quad 0 \leq \sigma_{m,s}^h \leq 1. \quad (4)$$

Letting s_h^f be the share of the f^{th} firm owned by the h^{th} household, and σ^f the profit of the f^{th} firm, the budget constraint for the h^{th} household is :

$$\sum_{m=k+1}^z p_m v_{m,s}^h r_{m,s}^h + \sum_{f=1}^F s_h^f \sigma^f \geq \sum_{n=1}^k p_n \varrho_{n,s}^h c_n^h, \quad \text{with probability unity,} \quad (5)$$

where the first term on the left gives the total income from the sale of factors perceived by the h^{th} household; the second term on the left gives the income from ownership perceived by the h^{th} household, and

the term on the right gives the total expenditure perceived by the h^{th} household. Thus the problem now is to maximize the expected utility for the h^{th} household

$$\max_{c_n^h, r_m^h} E[U^h] = \sum_{s=1}^S \pi(\xi) U^h(\varrho_s^h c^h, \sigma_s^h r^h), \quad (6)$$

subject to the budget constraint stated in (5).

The effective demand for the n^{th} good of the h^{th} household, and the effective supply of the m^{th} factor of the h^{th} household may be derived as functions of the prices and the market signals :

$$c_n^h = c_{n,s}^h(p, \xi), \quad r_m^h = r_{m,s}^h(p, \xi). \quad (7)$$

The aggregate effective demands for the n^{th} good, C_n^d and the aggregate effective supply of the m^{th} factor, R_m^s are, respectively, calculated by summing over households :

$$C_n^d = \sum_{h=1}^H c_n^h(p, \xi), \quad R_m^s = \sum_{h=1}^H r_m^h(p, \xi). \quad (8)$$

2.2 Firms

We write :

c_n^f is the f^{th} firm's effective supply of the n^{th} good, $n = 1, 2, \dots, k$,
 r_m^f is the f^{th} firm's effective demand for the m^{th} factor, $m = k+1, k+2, \dots, z$,

π_s is the probability-beliefs of an agent in the state s , $s = 1, 2, \dots, S$, $\pi_s = \pi_s(\xi) > 0$, and $\sum_{s=1}^S \pi_s(\xi) = 1$,

$\varrho_{n,s}^f$ is the rationing proportions to the f^{th} firm in the n^{th} good market in the state s ,

$\sigma_{m,s}^f$ is the rationing proportions to the f^{th} firm in the m^{th} factor market in the state s ,

$\varrho_{n,s}^f c_n^f$ is the actual trade in the n^{th} good market perceived by f^{th} firm in the state s ,

$\sigma_{m,s}^f r_m^f$ is the actual trade in the m^{th} factor market perceived by f^{th} firm in the states s ,

$\sigma_s^f r^f$ is the row vector of the actual trade in the m^{th} factor market perceived by f^{th} firm in the state s , $\sigma_{m,s}^f r_m^f$.

The profits of the f^{th} firm, σ^f , are revenue from sales less costs of purchases :

$$\sigma^f = \sum_{n=1}^k p_n q_{n,s}^f c_n^f - \sum_{m=k+1}^z p_m \sigma_{m,s}^f r_m^f. \quad (9)$$

We assume *voluntary exchange*, that is, the f^{th} firm is not forced to buy or sell more than his offers in all the markets and in all the states :

$$0 \leq q_{n,s}^f \leq 1, \quad 0 \leq \sigma_{m,s}^f \leq 1.$$

Each firm maximizes its expected value of the profits subject to the constraint of a production function, $f(\sigma^f r^f)$:

$$\max_{c_n^f, r_m^f} E[\sigma^f] = \sum_{s=1}^S \pi_s \left[\sum_{n=1}^k p_n q_{n,s}^f c_n^f - \sum_{m=k+1}^z p_m \sigma_{m,s}^f r_m^f \right], \quad (10)$$

$$\text{subject to } f(\sigma_s^f r^f) \geq \sum_{n=1}^k q_{n,s}^f c_n^f, \quad \text{with probability unity.} \quad (11)$$

The constraint dictates that the firm does not offer more than the output which can be produced from the least realized purchase from the factor markets.

The effective demand for the m^{th} factor and the effective supply of the n^{th} good may be derived as functions of the prices and the market signals :

$$r_m^f = r_m^f(p, \xi), \quad c_n^f = c_n^f(p, \xi). \quad (12)$$

The aggregate effective demand for the m^{th} factor R_m^d , and the aggregate effective supply of the n^{th} good, C_n^s respectively are :

$$R_m^d = \sum_{f=1}^F r_m^f(p, \xi), \quad C_n^s = \sum_{f=1}^F c_n^f(p, \xi). \quad (13)$$

3. Stochastic Rationing Mechanism

Let us use the m^{th} factor market to describe a stochastic rationing mechanism. We assume that the level of aggregate trade is the lesser of the aggregate effective demand and the aggregate effective supply, the so-called *short-side rule*. Formally :

$$\bar{R}_m = \min[R_m^d, R_m^s], \quad (14)$$

where \bar{R}_m is the actual trade. This condition imposes further restrictions on stochastic rationing probability distributions. In order to satisfy (14), the aggregate trade on the long side is no longer stochastic, but deterministic at the level of the short side. Although the trade level is deterministic, an agent on the long side faces stochastic trading. Formally, we have the following relationships :

If $\xi_m \geq 0$, then $\sigma_{m,s}^h = 1$ with probability 1.

If $\xi_m \leq 0$, then $\sigma_{m,s}^f = 1$ with probability 1.

Consider the h^{th} household offering r_m^h if $\xi_m \leq 0$. It perceives actual trade $\sigma_{m,s}^h r_m^h$ as random. For the sum of all households, the actual aggregate trade has to satisfy

$$\bar{R}_m = \sum_{h=1}^H \sigma_{m,s}^h r_m^h = R_m^d \quad \text{if } \xi_m \leq 0. \quad (15)$$

The f^{th} firm, which offers r_m^f , perceives the actual input $\sigma_{m,s}^f r_m^f$ as random, if $\xi_m \geq 0$. For the sum of all firms, the actual aggregate trade has to satisfy

$$\bar{R}_m = \sum_{f=1}^F \sigma_{m,s}^f r_m^f = R_m^s, \quad \text{if } \xi_m \geq 0. \quad (16)$$

Similarly, we can consider the stochastic rationing mechanism of the n^{th} good market. We assume the *short-side rule* on the actual trade in the n^{th} good market :

$$\bar{C}_n = \min[C_n^d, C_n^s], \quad (17)$$

where \bar{C}_n is the actual trade in the n^{th} good market. Since the aggregate trade on the long side is no longer stochastic, but deterministic at the level of the short side, we have the following relationships :

If $\xi_n \geq 0$, then $\varrho_{n,s}^f = 1$ with probability 1.

If $\xi_n \leq 0$, then $\varrho_{n,s}^h = 1$ with probability 1.

The actual trade amounts of the n^{th} good are formulated parallel to (15) and (16). Formally :

$$\bar{C}_n = \sum_{h=1}^H \varrho_{n,s}^h c_n^h = C_n^s \quad \text{if } \xi_n \geq 0. \quad (18)$$

$$\bar{C}_n = \sum_{f=1}^F \varrho_{n,s}^f c_n^f = C_n^d, \quad \text{if } \xi_n \leq 0. \quad (19)$$

The aggregate excess demands for the n^{th} good and the m^{th} factor are obtained by subtracting the aggregate effective supply from the aggregate effective demands :

$$E_n(p, \xi) = C_n^d(p, \xi) - C_n^s(p, \xi), \quad n = 1, 2, \dots, k. \quad (20)$$

$$E_m(p, \xi) = R_m^d(p, \xi) - R_m^s(p, \xi), \quad m = k+1, k+2, \dots, z. \quad (21)$$

We assume as follows,

Assumption 1 :

$$\frac{\partial E_i}{\partial \xi_j} \rightarrow 0, \quad \text{as } \xi \rightarrow 0, \quad (i, j = 1, 2, \dots, z). \quad (22)$$

where i and j denote either of a good or a factor.

If we assume that prices are announced and the market signals are expected by agents at discrete time intervals, then the rule that determines the adjustments in prices in each period can be stated as a discrete dynamic process. Let us consider :

$$\hat{p}_{i,t+1} = \hat{p}_{i,t} + \alpha_i F_i(\hat{p}_t, \xi_t), \quad i = 1, 2, \dots, z; \quad (23)$$

where $\hat{p}_t = \log p_t$ and $F_i(\hat{p}_t, \xi_t) = \psi_i[E_i(\exp \hat{p}_t, \xi_t)]$. $\psi_i[\cdot]$ is the price adjustment function for the i^{th} good (or factor) market that is assumed to sign-preserving function of the arguments, and α_i is the

adjustment coefficient that is positive. The price adjustment equations state that the rate of change of any price is an increasing function of excess demand for that commodity which decreases as excess demand decreases.

When the expectations of an agent are different from the actual market disequilibria, they revise their expectations adaptively and change their behavior in the next period. Then the process described above repeats itself. We may imagine the following dynamic process :

$$\xi_{i,t+1} = \xi_{i,t} + \beta_i G_i(p_t, \xi_t), \quad i = 1, 2, \dots, z; \quad (24)$$

where $G_i(\hat{p}_t, \xi_t) = \phi_i[E(\exp \hat{p}_t, \xi_t) - \xi_t]$, $\phi_i[\cdot]$ is the expectations adjustment function for the i^{th} good (or factor) market that is assumed to sign-preserving function of the arguments, and β_i is the quantity adjustment coefficient for the disequilibrium signal that is a positive constant.

The expectations adjustment equations state that the expected value of the market signal is raised if the actual value of the market disequilibrium is higher than the expected value.

For convenience of analysis, we rewrite the adjustment equations (23) and (24) as follows ;

$$X_{t+1} = X_t + \zeta H(X_t), \quad X_t \in R^{2z}, \quad (25)$$

where

$$X_t \equiv \begin{bmatrix} \hat{p}_t' \\ \xi_t' \end{bmatrix} \quad \zeta H(X_t) \equiv \begin{bmatrix} \alpha F(\hat{p}_t, \xi_t) \\ \beta G(\hat{p}_t, \xi_t) \end{bmatrix} \quad (26)$$

where \hat{p}_t' and ξ_t' denote the column vectors of $\hat{p}_{i,t}$ and $\xi_{i,t}$ respectively and α is the diagonal matrix of α_i , and β the diagonal matrix of β_i , and $F(\hat{p}_t, \xi_t)$ and $G(\hat{p}_t, \xi_t)$ denote the column vectors of $F_i(\hat{p}_t, \xi_t)$ and $G_i(\hat{p}_t, \xi_t)$, ($i = 1, 2, \dots, z$) respectively.

4 Existence of Multiple Equilibria

In the foregoing section we specialize a disequilibrium dynamic process. This section is concerned with the existence of equilibria in the foregoing dynamic system (25). To prove the existence of equilibria, we use the well-known *Poincaré-Hopf theorem*. The equilibria

are pairs of $X^* = (\hat{p}^*, \xi^*)$ if (i) actual excess demands for inputs and outputs equal zero so the prices and wages level \hat{p} is unchanging ; (ii) the vector of the market signals ξ equals actual excess demands so ξ are unchanging. Therefore, in all equilibria each market signal ξ must be equal to zero, that is, $X^* = (\hat{p}^*, 0)$.

Let us assume that the state space of the economy defined above can be represented by a manifold with boundary, and the economic system is *self correcting* for extreme disequilibria : as some price (or some wage) goes to the lower (upper) boundary of the state space of economy, its excess demand becomes positive (negative). As some quantity signal goes to the boundary, the economic system becomes strong enough to push the system away from that boundary. We define H as a continuous vector field which is composed of the functions $F_1(\hat{p}, \xi), F_2(\hat{p}, \xi), \dots, F_z(\hat{p}, \xi), G_1(\hat{p}, \xi), G_2(\hat{p}, \xi), \dots, G_z(\hat{p}, \xi)$. We assume as follows :

Assumption 2 (the boundary conditions) : (i) The state space of the economy is a compact convex multidimensional smooth manifold with a boundary which we will denote by M . H are continuous vector field on M .

(ii) The vector field H points inward on the boundary of M .

(iii) The economy defined by H is regular.

On the Jacobian determinant of the excess demand functions F with respect to prices we assume further as follows :

Assumption 3 :

$$(-1)^z \det(DF(\hat{p}^*, \xi^*)) < 0, \quad \text{at any equilibrium.} \quad (27)$$

where $\det(DF(\hat{p}^*, \xi^*))$ denotes the Jacobian determinant of $F_1(\hat{p}^*, \xi^*), F_2(\hat{p}^*, \xi^*), \dots, F_z(\hat{p}^*, \xi^*)$ with respect to $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_z$.

We can get the following theorem :

Theorem 1 : If Assumption 1, 2 and 3 hold, then there must be multiple equilibria of the economic system.

Proof : From Assumption 2, $-H$ defines a smooth vector field on the disk which points outwards on the boundary of M and has a finite number of isolated equilibria. Since

$$\det(-DH(\hat{p}^*, \xi^*)) = (-1)^z \prod_{i=z}^z \frac{\partial G_i}{\partial \xi_i} \det(DF(\hat{p}^*, \xi^*)), \quad (28)$$

from Assumption 3, the Jacobian determinant,

$$\det(-DH(\hat{p}^*, \xi^*))$$

evaluated at a equilibrium $X^* = (\hat{p}^*, \xi^*)$ is negative. By the *Poincaré-Hopf theorem*³, while the index of the equilibrium X^* is -1 , the summation of the indices of each equilibrium must be $+1$. Therefore, Theorem 1 holds. Q.E.D.

We obtain Theorem 2 as its corollary.

Theorem 2: Assume that

$$(-1)^z \det(DF(\hat{p}^*, \xi^*)) > 0 \quad \text{at all equilibria.} \quad (29)$$

Under the assumption (29), there exist a unique equilibrium, $X^* = (\hat{p}^*, \xi^*)$.

Its proof is omitted since it follows from Poincaré-Hopf theorem.

5. Dynamics

In this section we investigate the dynamical propensities of disequilibrium dynamics.

bigskip 5.1 Stability of the equilibrium

To begin with we demonstrate a necessary and sufficient condition for the Walrasian equilibrium to be locally stable.

Theorem 3: Assume that

- (i) $1 + \alpha_i DF_{ii} > 0$, and $DF_{ij} > 0$ ($i \neq j$).
- (ii) $-2 < \beta_i DG_{ii}$.
- (iii) (Hicks conditions for perfect stability)⁴

$$DF_{ii}, \quad \left| \begin{array}{cc} DF_{ii} & DF_{ij} \\ DF_{ji} & DF_{jj} \end{array} \right|, (j > i), \quad \left| \begin{array}{ccc} DF_{ii} & DF_{ij} & DF_{il} \\ DF_{ji} & DF_{jj} & DF_{jl} \\ DF_{li} & DF_{lj} & DF_{ll} \end{array} \right|, (l > j > i),$$

....., $|DF|$

where $DF_{ij} = \partial^2 F(\hat{p}_t, \xi_t) / \partial \hat{p}_i \partial \hat{p}_j$, $(i, j = 1, 2, \dots, z)$, are alternately negative and positive. Under the assumptions there is a unique Walrasian equilibrium $X^* = (\hat{p}^*, \xi^*)$ and the Walrasian equilibrium is locally stable.

Proof:

From theorem 2 it follows that there is a unique Walrasian equilibrium. The stability of the equilibrium requires that the moduli of all the characteristic roots of (25) are less than unity.

For simplicity of analysis let us rewrite the disequilibrium dynamics (25) as follows :

$$X_{t+1} = W_\zeta(X_t), \quad X_t \in R^{2z} \quad (30)$$

where $W(X_t, \zeta) \equiv X_t + \zeta H(X_t)$. The Jacobian matrix of $W_\zeta(X^*)$ with respect to the equilibrium X^* is as follows,

$$DW_\zeta(X^*) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \quad (31)$$

where A , B , C and D are the square submatrices which have order $z \times z$. A denotes the Jacobian matrix of $\hat{p}_i^* + \alpha_i F_i(\hat{p}^*, \xi^*)$, with respect to \hat{p}_i^* , $(i = 1, 2, \dots, z)$, B the Jacobian matrix of $\hat{p}_i^* + \alpha_i F_i(\hat{p}^*, \xi^*)$, with respect to ξ_i^* , $(i = 1, 2, \dots, z)$, C the Jacobian matrix of $\xi_i^* + \beta_i G_i(\hat{p}^*, \xi^*)$, with respect to \hat{p}_i^* , $(i = 1, 2, \dots, z)$, and D the Jacobian matrix $\xi_i^* + \beta_i G_i(\hat{p}^*, \xi^*)$, with respect to ξ_i^* , $(i = 1, 2, \dots, z)$.

From Assumption 1, B is a null matrix. It means that $DW(X^*)$ is decomposable. Therefore the Jacobian determinant of $DW(X)$ is

$$\det(DW_\zeta(X)) = \left| \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right| = |A||D|. \quad (32)$$

It means that the proof can be divided into two parts. First, it will be shown that all the characteristic roots of the matrix D have moduli less than 1. Second it will be shown that the moduli of all characteristic roots of the matrix A are less than unity.

D is the diagonal matrix $\text{diag}(\beta_1 \frac{\partial G_1}{\partial \xi_1}, \beta_2 \frac{\partial G_2}{\partial \xi_2}, \dots, \beta_z \frac{\partial G_z}{\partial \xi_z})$. Since $-2 < \beta_i DG_{ii}$, all the characteristic roots of the diagonal matrix B have moduli less than 1. This completes the first part of our proof.

By assumption 3 (i), A can be made non-negative. By the Frobenius theorem on non-negative matrices, A has the Frobenius root $\rho(A)$ and $\rho(A)$ is greater than the moduli of all characteristic roots of A . Therefore if $\rho(A)$ is less than unity, then all characteristic roots of A are less than unity.

The characteristic equation of the matrix A is

$$\begin{vmatrix} 1 + \alpha_1 DF_{11} - \lambda & \alpha_1 DF_{12} & \dots & \alpha_1 DF_{1z} \\ \alpha_2 DF_{21} & 1 + \alpha_2 DF_{22} - \lambda & \dots & \alpha_2 DF_{2z} \\ \dots & \dots & \dots & \dots \\ \alpha_z DF_{z1} & \alpha_z DF_{z2} & \dots & 1 + \alpha_z DF_{zz} - \lambda \end{vmatrix} = 0. \quad (33)$$

where λ is a characteristic root of A . In its expanded form, this equation may be written

$$\begin{aligned} \rho^z &- \sum \alpha_i DF_{ii} \rho^{n-1} + \sum \alpha_i \alpha_j \rho^{n-2} \left| \begin{vmatrix} \alpha_1 DF_{ii} & \alpha_1 DF_{ij} \\ \alpha_2 DF_{ji} & \alpha_2 DF_{jj} \end{vmatrix} \right| \\ &+ \dots + (-1)^z \alpha_1 \alpha_2 \dots \alpha_z |DF| = 0. \end{aligned} \quad (34)$$

where $\rho = (\lambda - 1)$. Since the α_i are all positive, we know by Descartes' rule of signs that (34) can have no positive real roots equal to or greater than unity if the determinants,

$$DF_{ii}, \left| \begin{vmatrix} DF_{ii} & DF_{ij} \\ DF_{ji} & DF_{jj} \end{vmatrix} \right|, (j > i), \left| \begin{vmatrix} DF_{ii} & DF_{ij} & DF_{il} \\ DF_{ji} & DF_{jj} & DF_{jl} \\ DF_{li} & DF_{lj} & DF_{ll} \end{vmatrix} \right|, (l > j > i), \dots, |DF|$$

are alternately negative and positive. This completes proof of dynamic stability of the equilibrium, X^* . Q.E.D.

5.2 Chaotic dynamics

Then we demonstrate a sufficient condition for the discrete-time disequilibrium dynamic process to lead to chaos. Since agents cannot

realize their desired transactions in our model, actual transactions are generally different from the expected transactions. These disequilibria in the markets cause macroeconomic dynamics. If the economic system has multiple equilibria, then we can arrive at the following important result :

Theorem 4 : dynamic If Assumptions 1 and 2 hold, then there exist positive numbers of the adjustment coefficients α^* and β^* such that the disequilibrium dynamic process (25) is chaotic for any α_i satisfying $\alpha_i > \alpha^*$ and any β_i satisfying $\beta_i > \beta^*$ for $(i = 1, 2, \dots, z)$.

Proof : The theorem can be proven easily utilizing Theorem 1 and Hata's theorem. [See Hata (1982)]. Now we describe the proof of the above theorem.

Let us rewrite the disequilibrium dynamic process (25) as follows :

$$X_{t+1} = W_{\zeta}(X_t), \quad X_t \in R^{2z} \quad (35)$$

where $W_{\zeta}(X_t) \equiv X_t + \zeta H(X_t)$. From Theorem 1 there are at least two equilibria, X^* and X^{**} in the equation (35).

Let $B(X, r)$ denote the closed ball in R^{2z} of radius r centered at X .

Before proving theorem 2 we shall present preliminary lemmas. See Hata (1982) about the proof of lemmas.

Lemma 4.1 : There exist $r_1 > 0$ and $\zeta_1 > 0$ such that $\det DW_{\zeta}(X) \neq 0$ for any $\zeta > \zeta_1$ and any $X \in B(X^*, r_1) \cup B(X^{**}, r_1)$.

Lemma 4.2 : There exist $r_2 > 0$ and $\lambda_2 > 0$ such that all eigenvalues of $\det DW_{\zeta}(X)$ exceed unity in norm for any $\zeta > \zeta_2$ and any $X \in B(X^*, r_2)$.

Lemma 4.3 : For a sufficiently small open neighborhood U of X^* and any bounded set V , there exists $\zeta_3(U, V) > 0$ such that the equation $W_{\zeta}(u) = v$ has at least one solution $u \in U$ for any $\zeta > \zeta_3(U, V)$ and any $v \in V$. Note that the similar arguments hold for X^{**} .

Now we are ready for the proof of the theorem. Select sufficiently small open neighbourhoods U, V of X^*, X^{**} respectively such

that $U \cap V = \emptyset$ and Lemma 4.3 holds for both X^* and X^{**} . Let $r^* = \min(r_1, r_2)$ and $\zeta^* = \max(\zeta_1, \zeta_2, \zeta_3(U, V), \zeta_3(V, U))$. Without loss of generality we can assume that $U \subset B(X^*, r^*)$ and $V \subset B(X^{**}, r^*)$. By Lemma 4.3, for any $\zeta > \zeta^*$, there exist $v_\zeta \in V$ and $u_\zeta \in U$ such that $W_\zeta(v_\zeta) = X^*$ and $W_\zeta(u_\zeta) = v_\zeta$. Since $\det DW_\zeta(u_\zeta) \neq 0$ and $\det DW_\zeta(v_\zeta) \neq 0$ by Lemma 4.1, we can find $r_\zeta > 0$ such that $B(u_\zeta, r_\zeta) \subset B(X^*, r^*)$, $W_\zeta(B(u_\zeta, r_\zeta)) \subset V$ and $W_\zeta^2(B(u_\zeta, r_\zeta)) \subset U$, and both $W_\zeta|_{B(u_\zeta, r_\zeta)}$ and $W_\zeta|_{W_\zeta(B(u_\zeta, r_\zeta))}$ are homeomorphisms. Finally define compact sets $\{B_k\}_{-\infty < k \leq 2}$ as follows :

$$B_1 = W_\zeta(B(u_\zeta, r_\zeta)), \quad B_2 = W_\zeta^2(B(u_\zeta, r_\zeta))$$

and $B_{-k} = W_\zeta^{-k}(B(u_\zeta, r_\zeta))$ for $k \geq 0$, since W_ζ^{-k} is well-defined by Lemma 4.2. This shows that X^* is a snap-back repeller. Obviously the same argument holds for X^{**} . Marotto (1978) showed that multidimensional discrete-time systems are chaotic if they have the snapback repeller. Hence the completion of the proof. Q.E.D.

6 Concluding Remarks

This paper demonstrates a necessary and sufficient condition for dynamic stability of a unique Walrasian equilibrium, and a sufficient condition for disequilibrium dynamics, to lead to chaos in a general disequilibrium model with stochastic rationing and price flexibility. In conclusion, (i) If Hicks conditions of perfect stability holds, and the adjustment speeds are sufficiently slow, there is a unique equilibrium and the equilibrium is locally stable. (ii) If there are multiple equilibria, and both of the adjustment speeds of prices and quantities are sufficiently fast, the disequilibrium dynamic process is chaotic, in the sense of Li-Yorke.

The foregoing results suggest that the propensities of disequilibrium dynamics depend on the combinations of the adjustment speeds and the number of equilibria. What has to be noticed is that the existence of multiple equilibria is not a necessary condition, but one of the sufficient conditions for the disequilibrium dynamics to lead to chaos. The occurrence of chaos means that the dynamic structure is inherently nonlinear. While the existence of multiple equilibria is sufficient for the nonlinearity of the dynamic system, it is not always necessary. Thus it remains an unanswered question what are necessary and sufficient conditions for chaotic dynamics, it needs further

consideration. We also need to extend the dynamic analysis by using an intertemporal model of the household and the firm, that is, firms may carry over inventories from one period to the next, just as households may hold money balances⁵.

In recent years several attempts have been made by scholars to show whether there were empirical evidence of deterministic chaos of macroeconomic variables [Brock and Sayers (1988), Frank, Gencay and Stengos (1988) and Frank and Stengos (1988)]. So far there is not enough evidence to decide the matter, but it seems likely that irregular fluctuations of many macroeconomic variables are chaotic. Our results may offer the key to an theoretical explanation of why the time series of many macroeconomic variables have the appearance of having irregular fluctuations.

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Notes

¹The formulation of out-of-equilibrium transactions in the non-tâtonnement literature [Negishi (1961) and Hahn and Negishi (1962)] prefigured some aspects of non-Walrasian analysis.

²The Hata theorem generalizes Yamaguti-Matano theorem [Yamaguti and Matano (1979)]. Previous paper [Kaizoji (1994)] demonstrated sufficient conditions for the discrete-time tâtonnement process to lead to chaos in the competitive economy with two commodities by applying Yamaguti-Matano theorem.

³For a clear exposition and proof of the Poincaré-Hopf theorem see Milnor (1972), Varian (1975), and Mas-Colell (1985).

⁴See Metzler (1945) for a full account of Hicks conditions.

⁵For an example of the an intertemporal model of the firm and household see Muellbauer and Portes (1978).

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複数均衡と不均衡動学

〈要 約〉

海蔵寺 大成

過去数十年の間、一般均衡モデルの均衡存在とその価格調整プロセスに関して多くの研究が積み重ねられてきた。これは、市場の価格調整メカニズムが自然に経済システムを均衡状態に導くという基本的アイデアを多くの経済学者が受け入れていたためである。しかし、現実の経済では、需要と供給が乖離している場合、価格による調整だけでなく、数量による調整が同時に行われるのが普通である。それにもかかわらず、現在まで価格と数量が同時に調整されて行くメカニズム、いわゆる不均衡動学の研究は十分行われてきたとはいえない。

この論文の目的は価格調整と数量調整が同時に働くモデル(一般不均衡モデル)を構築し、その不均衡動学プロセスの性質を調べることである。我々の主要な結論はつぎの二つである。(1) 経済システムが唯一の均衡点を持ち、かつ価格調整と数量調整の速度が十分に遅い場合、経済システムは均衡点に収束して行く。(2) 経済システムに複数の均衡点が存在し、かつ価格調整と数量調整の速度が早い場合、価格と数量はカオス的な変動を起こし、均衡状態は達成されない。