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Stability of Perfect Foresight Equilibria with Adaptive Learning Rules

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Abstract

The aim of this paper is to demonstrate that an adaptive learning rule can lead the economic system to perfect foresight (or rational expectations) equilibrium in a Samuelson-type overlapping generations model of a monetary economy.

1 Introduction

Research questions on stability of perfect foresight (or rational expectations) equilibria, what is called *expectational stability as a equilibrium selection criterion*, are asked increasingly often in the recent literature, such as whether simple and plausible learning rules exist that do converge to perfect foresight equilibria under general conditions, and moreover, whether such learning rules can single out a perfect foresight equilibrium, corresponding to a monetary steady state. Over the last few decades a large number of studies have been made on expectational stability, originally proposed by Decanio (1979), Friedman (1979) and Lucas (1978), and extensively developed by Bray (1982), Bray and Savin (1986), Evans (1983,1985,1986,1989), and Evans and Honkapohja (1994a)[1].

In this paper attention is focused on stability of perfect foresight equilibria with adaptive learning rules (or adaptive expectations) in a Samuelson-type overlapping generations model of monetary economy. Adaptive learning rules in overlapping generations models have been studied often in the literature on learning. See Lucas (1986), Guesnerie and Woodford (1991), and Evans and Honkapohja (1995). Lucas (1986) is an early example of the analysis of adaptive learning as a way of selecting among the multiple possible equilibria in an overlapping generations model. He has suggested that the outcome of an adaptive learning process in the overlapping generations model will be a perfect foresight equilibrium.

The aim of this paper is to demonstrate Lucas's conjecture. Our main result is that an adaptive learning rule leads the economy to the perfect foresight equilibrium, pr ovided that there are two steady states, corresponding to the perfect foresight equilibrium[2], and the autarchy steady state, corresponding to no exchange between generations. For simplicity we will call the stationary perfect foresight equilibrium the perfect foresight equilibrium below.

[1] In linear dynamic economic models the issue of convergence of least squared learnings to rational expectations equilibria has been studied by Marcet and Sargent (1988,1989). On the other hand learning in nonlinear models has been considered by Bullard (1994), Evans and Honkapohja (1994b,1995), Fuchs (1979), Grandmont (1985), Grandmont and Laroque (1986,1990,1991), Woodford (1990), Guesnerie and Woodford (1991), and Kaizoji (1994).

[2] The steady state is often called the stationary monetary equilibrium.

2 The model

The model involves one non-storable good and a fiat money, that is employed for transferring wealth from one period to the next. The money stock, M will be assumed to be constant over time. The model does not bequest. For simplicity, production is taken to be exogenous. Agents live two periods and are identical (equivalently there is a single agent) in each generation. The agents' endowments of the good at each age $\tau = 1, 2$, are $l_1^* > 0$, $l_2^* > 0$. The agent's tastes among consumption streams $a_1 \geq 0$, $a_2 \geq 0$ are described by the separable utility function $V_1(a_1) + V_2(a_2)$.

We shall assume as follows :

Assumption 1 : For each $\tau = 1, 2$, $V_\tau(a_\tau)$ is continuous on $[0, +\infty)$ and twice continuously differentiable on $(0, +\infty)$. Moreover, $V_\tau'(a_\tau) > 0$, $V_\tau''(a_\tau) < 0$ for a_τ, and $\lim_{a_\tau \to 0} V_\tau'(a_\tau) = +\infty$, $\lim_{a_\tau \to +\infty} V_\tau'(a_\tau) = 0$.

We focus on the *Samuelson case*, that is,

Assumption 2 : $\bar{\theta} = V_1'(l_1^*)/V_2'(l_2^*) < 1$.

Let us define the so-called *Arrow-Pratt relative degree of risk aversion* as $R_\tau(a_\tau) = -V_\tau''(a_\tau)a_\tau/V_\tau'(a_\tau)$ which are well defined whenever $a_\tau > 0$. We shall use the following assumption :

Assumption 3 : $R_2(a_2)$ is a nondecreasing function of a_2 for every $a_2 > 0$.

Under the foregoing environment we can get the well-defined difference equation that represents the law of motion of the economic system

$$V_1'(l_1^* - \mu_t)\mu_t = V_2'(l_2^* + \mu_{t+1}^e)\mu_{t+1}^e, \tag{1}$$

where μ_t and μ_{t+1}^e denote the real balances and the real balances expected by the young agent, respectively.

If we write $v_1(\mu_t) = \mu_t V_1'(l_1^* - \mu_t)$ and $v_2(\mu_{t+1}^e) = \mu_{t+1}^e V_2'(l_2^* + \mu_{t+1}^e)$, then

$$\mu_t = v_1^{-1}[v_2(\mu_{t+1}^e)] \equiv \chi(\mu_{t+1}^e). \tag{2}$$

The function $\chi(\mu)$ is continuously differentiable on the open interval $(0, +\infty)$. We see that v_1 is a differentiably increasing function that maps the interval

$[0, l_1^*)$ onto $[0, +\infty)$, while v_2 maps $[0, +\infty)$ into itself. One has $\chi(\mu^e)$ for all $\mu^e \geq 0$. If $\rho_1 = \sup R_2(a_2) \leq 1$, $\chi(\mu^e)$ is increasing everywhere, while $\chi(\mu^e)$ has a unique maximum under Assumption 3 when $\rho_2 > 1$. Therefore *under Assumptions 1,2 and 3 there is a unique perfect foresight equilibrium μ^*.* For further details see Grandmont (1985).

2.1 Adaptive Learning Dynamics

We have a dynamic system where the expectations determine the current variables through equation (2). In order to define the dynamics fully, we need to specify the expectations formation function. If agents do know at the beginning of the date t only the current and past values of the real balances, $(\mu_t, \mu_{t-1}, \mu_{t-2}, \ldots\ldots\ldots)$, then young agents have to forecast the future values of the real balance by using the current and past values of the real balances at the beginning of the date t. One way to model expectations formation is to postulate as in Grandmont and Laroque (1986), that at each date, the real balance μ_{t+1}^e expected by young agents forecast at the beginning of the date t is a fixed function of the current and past values of the real balance $(\mu_t, \mu_{t-1}, \mu_{t-2}, \ldots\ldots\ldots\ldots)$. We assume that the expectations function is formed as *the weighted average of past values.*

$$\mu_{t+1}^e = \alpha \sum_{T=1}^{\infty} (1-\alpha)^{T-1} \mu_{t-T}, \quad 0 < \alpha < 1.$$

We can rewrite the above equation the following

$$\mu_{t+1}^e = \mu_t^e + \alpha[\mu_t - \mu_t^e], \quad 0 < \alpha < 1. \tag{3}$$

where α is a constant of proportionality called the expectations coefficient. With the adaptive learning rule (3), expectation is revised acoording to the gap between previous expectations and realizations. Substituting (2) into (3), we get the following

$$\mu_{t+1}^e = (1-\alpha)\mu_t^e + \alpha\chi(\mu_{t+1}^e). \tag{4}$$

The difference equations (2) and (4) define a temporary monetary equilibrium dynamics with adaptive expectations, that is, *adaptive learning dynamics.*

Lemma 1 : Under Assumptions 1 and 2, a steady state in adaptive learning dynamics is equal to a perfect foresight equilibrium.

Proof : If a steady state of the equation (2) is equal to a steady state of the difference equation (4), then Lemma 1 is proved. The proof is immediate. The perfect foresight equilibrium satisfies the equation $\mu^* - \chi(\mu^*) = 0$. Then the equation (4) also becomes the steady state $\mu^e_{t+1} = \mu^e_t$. Thus the above statement is proved. (Q.E.D.)

We will investigate the properties of the forward temporary equilibrium dynamics in Section 3 below.

3 Stability of Perfect Foresight Equilibria

In the present section we study stability for a perfect foresight equilibrium in adaptive learning dynamics which are generated by the equations (2) and (4).

For convenience of analysis we rewrite (4) as follows :

$$F(\mu^e_{t+1}, \mu^e_t) \equiv \mu^e_{t+1} - \alpha\chi(\mu^e_{t+1}) - (1-\alpha)\mu^e_t = 0. \tag{5}$$

Since the function $\chi(\mu^e_{t+1})$ is continuously differentiable on the open interval $(0, +\infty)$, $F(\mu^e_t, \mu^e_{t+1})$ has continuous partial derivatives on a neighborhood of an equilibrium (μ^*, μ^*) at which $F(\mu^*, \mu^*) = 0$ and $F_{\mu^e_{t+1}}(\mu^*, \mu^*) \neq 0$ where $F_{\mu^e_{t+1}}$ denotes $\partial F / \partial \mu^e_{t+1}$. Under the above conditions we can apply the *implicit function theorem* to the form $F(\mu^e_t, \mu^e_{t+1}) = 0$. The implicit function theorem states that if F is smooth and if a equilibrium point, (μ^*, μ^*), is a point at which $F_{\mu^e_{t+1}}$ does not vanish, then it is possible to express μ^e_{t+1} as a function of μ^e_t in a region containing the equilibrium point. In other words we can define forward adaptive learning dynamics $\mu^e_{t+1} = g(\mu^e_t)$ in a region containing the equilibrium point, (μ^*, μ^*). Furthermore, the derivative of g is given by the implicit differentiation formula[3] :

[3]Tarui (1991) has originally pointed out that (6) is a condition of a local stability for a steady state.

$$g'(\mu^*) = -\frac{F_{\mu_t^e}(\mu^*, \mu^*)}{F_{\mu_{t+1}^e}(\mu^*, \mu^*)}. \tag{6}$$

From the foregoing analysis we can derive the following

Theorem 1 : Let $\bar{\mu}^*$ be the perfect foresight equilibrium which has the least value among the multiple possible equilibria. If Assumption 1 and 2 hold, then the perfect foresight equilibrium $\bar{\mu}^*$ is locally stable.

Proof : we get the following

$$g'(\bar{\mu}^*) = -\frac{F_{\mu_t^e}(\bar{\mu}^*, \bar{\mu}^*)}{F_{\mu_{t+1}^e}(\bar{\mu}^*, \bar{\mu}^*)} = \frac{(1-\alpha)}{1-\alpha\chi'(\bar{\mu}^*)}. \tag{7}$$

Since $\chi'(\bar{\mu}^*)$ is always less than 1 in the Samuelson case, $g'(\bar{\mu}^*)$ is always less than 1. This means that forward adaptive learning dynamics $\mu_{t+1}^e = g(\mu_t^e)$ are locally stable. It follows from Lemma 1 that $\bar{\mu}^*$ is locally stable. (Q.E.D.)

Next we look at a condition of global stability of a perfect foresight equilibrium.

Theorem 2 : If Assumptions 1, 2 and 3 hold, the unique perfect foresight equilibrium μ^* that is globally stable exists.

Proof : We define the following

$$\mu_t^e = \frac{1}{(1-\alpha)}\mu_{t+1}^e - \frac{\alpha}{(1-\alpha)}\chi(\mu_{t+1}^e) \equiv f(\mu_{t+1}^e). \tag{8}$$

Since $f(\mu_{t+1}^e)$ is continuous and monotonically increasing, or $f(\mu_{t+1}^e)$ has a single minimum value, it follows that the map $f(\mu_{t+1}^e)$ is a continuous and monotonically increasing function which has an interval $I = [a, b]$ for domain where a is a positive value of $f(\mu_{t+1}^e) = 0$, and has range $J = [0, d]$. It is possible that b and d are infinite. (See Figures 1 and 2.) Then, there is the inverse function $g(\mu_t^e)$ of $f(\mu_{t+1}^e)$, (that is, $\mu_{t+1}^e = g(\mu_t^e)$) with domain J.

Since the map f has the unique perfect foresight equilibrium μ^*, f satisfies the following

Condition 1 :

$$f(0) = 0, \quad f(\mu^*) = \mu^*,$$
$$f(\mu^e_{t+1}) < \mu^e_{t+1} \qquad \text{for } a < \mu^e_{t+1} < \mu^*,$$
$$f(\mu^e_{t+1}) > \mu^e_{t+1} \qquad \text{for } \mu^* < \mu^e_{t+1} < b.$$

Therefore the inverse function g satisfies the following conditions

Condition 2 :

$$g(0) = 0, \quad g(\mu^*) = \mu^*,$$
$$g(\mu^e_t) > \mu^e_t \qquad \text{for } a < \mu^e_t < \mu^*,$$
$$g(\mu^e_t) < \mu^e_t \qquad \text{for } \mu^* < \mu^e_t < d.$$

It follows from Condition 2 that $g(\mu^e_t)$ is a *contraction mapping* on R_+. (See Figure 3.) Thus a perfect foresight equilibrium μ^* is globally stable from the *contraction mapping theorem.* (Q.E.D.)

3.1 Example

Suppose that the utility function for each generation is given by

$$V_\tau(a_\tau) \doteq a_\tau^{(1-\beta_\tau)}/(1-\beta_\tau), \quad \tau = 1, 2. \tag{9}$$

The utility function (12) satisfies Asumptions 1, 2 and 3. Therefore there exists a unique perfect foresight equilibrium μ^*. As it concerns the atitude to risk, the function is known as constant relative risk aversion (CRRA), where the RRA coefficient is - $V"_\tau(a_\tau)/V'(a_\tau) = \beta_\tau$. We assume that $\beta_1 = 1$ and $\beta_2 > 1$. Then $F(\mu^e_{t+1}, \mu^e_t)$ is as follows :

$$F(\mu^e_{t+1}, \mu^e_t) \equiv \frac{1}{(1-\alpha)}\mu^e_{t+1} - \frac{\alpha}{(1-\alpha)} l_1^* \mu^e_{t+1}/[\mu^e_{t+1} + (l_2^* + \mu^e_{t+1})^\beta] - \mu^e_t = 0, \tag{10}$$

$$g'(\mu^*) = \frac{(1-\alpha)}{1-\alpha(1-\mu^*(1+\beta(l_2^*+\mu^*)^{\beta_2-1}/l_1^*)}. \tag{11}$$

Since $\mu^*(1+\beta_2(l_2^*+\mu^*)^{\beta_2-1}/l_1^* > 0$, we get $g'(\mu^*) < 1$. The foregoing result prove that a perfect foresight equilibrium is locally stable.

For analysis of the global stability of μ^* we rewrite (13) as follows :

$$\mu_t^e = \frac{1}{(1-\alpha)}\mu_{t+1}^e - \frac{\alpha}{(1-\alpha)} l_1^* \mu_{t+1}^e / [\mu_{t+1}^e + (l_2^* + \mu_{t+1}^e)^\beta] \equiv \tilde{f}(\mu_{t+1}^e). \quad (12)$$

The function $\tilde{f}(\mu_{t+1}^e)$ is continuous and has a single minimum value. Furthermore $\tilde{f}(\mu_{t+1}^e)$ satisfies Conditions 1 and 2. Therefore it follows from Theorem 2 that μ^* is grobally stable.

4 Concluding Remarks

In this paper we demonstrate that adaptive expectations always lead the dynamic system to the perfect foresight equilibria in a Samuelson-type overlapping generations model of a monetary economy. Once forward adaptive learning dynamics converge to a perfect foresight equilibrium, a generation's learning becomes complete, so that the generation has perfect foresight. New generations who are born subsequently face the situation that is exactly the same, in all respects, as that faced by the preceding generation. Thus they also have perfect foresight. This conclusion is independent of the agent's degree of risk aversion. It is known from Benhabib and Day (1982), and Grandmont (1985), and other authors that backward dynamics may be chaotic, provided that agents have perfect foresight (that is, $\mu_{t+1} = \mu_t$) and are very risk averse. Consider the constant elasticity case, that is, $V_\tau(a_\tau) = a_\tau^{1-\beta_\tau}/(1-\beta_\tau)$ once again. Backward perfect foresight dynamics are chaotic in the sense of Li-Yorke, as the Corollary of Grandmont (1985) demonstrates, when β_2 is large enough provided that the other conditions are constant. On the contrary we can illustrate that the larger the old generation's RRA coefficient, β_2 becomes, the more rapidly forward adaptive learning dynamics converge to the perfect foresight equilibrium. To put it another way, the more the old generation becomes risk averse, the more rapidly the learning is complete.

On these grounds we have come to the conclusion that an agent's adaptive behavior does lead to perfect foresight (or rational expectations) equilibrium under general conditions, and therefore there is considerable justification for the assumption of rational expectations in the long run.

References

[1] Bullard, J. (1994), "Learning Equilibria," *Journal of Economic Theory*, 64, 468-485.

[2] Bray, M. (1982), "Learning, Estimation and the Stability of Rational Expectations Equilibria," *Journal of Economic Theory*, 26, 318-339.

[3] Bray, M. and N. E. Savin (1986), "Rational Expectations Equilibria, Learning, and Model Specification," *Econometrica*, 54, 1129-1160.

[4] DeCanio, S. J. (1979), "Rational Expectations and Learning from Experience," *Quarterly Journal of Economics*, 93, 47-58.

[5] Evans, G. W. (1983), "The Stability of Rational Expectations in Macroeconomic Models", in R. Frydman and E. S. Philips (eds.) *Individual Forecasting and Aggregate Outcomes*, Cambridge: Cambridge University Press.

[6] Evans, G. W. (1985), "Expectational Stability and the Multiple Equilibria Problem in Rational Expectations Models", *Quarterly Journal of Economics*, 100 : 1217-33.

[7] Evans, G. W. (1986), "Selection Criteria for Models with Non-Uniqueness", *Journal of Monetary Economics*, 18 : 147-57.

[8] Evans, G. W. (1989), "The Fragility of Sunspots and Bubbles," *Journal of Monetary Economics*, 23 : 297-317.

[9] Evans, G. W. and S. Honkapohja, (1992), "On the Robustness of Bubbles in Linear RE Models," *International Economic Review*, 33, 1-14.

[10] Evans, G. W. and S. Honkapohja, (1994a), "Learning, Convergence, and Stability with Multiple Rational Expectations Equilibria," *European Economic Review*, 38, 1071-1098.

[11] Evans, G. W. and S. Honkapohja, (1994b), "On Local Stability Sunspot Equilibria under Adaptive Learning Rules," *Journal of Economic Theory*, 64, 142-161.

[12] Evans, G. W. and S. Honkapohja, (1995), "Local Convergence of Recursive Steady States and Cycles in Stochastic Nonlinear Models," *Econometrica*, 63, 195-206.

[13] Friedman, B. M. (1979) "Optimal Expectations and the Extreme Information Assumptions of Rational Expectations Macro Models," *Journal of Monetary Economics*, 5, 23-41.

[14] Fuchs, G. (1977), "Formation of Expectations : A model in temporary general equilibrium theory," *Journal of Mathematical Economics*, 4, 167-187.

[15] Fuchs, G. (1979), "Is Error Learning Behavior Stabilizing ?," *Journal of Ecomonic Theory*, 20, 300-317.

[16] Grandmont, J. M. (1985) "On Endogenous Competitive Business Cycles," *Econometrica*, 53, 995-1045.

[17] Grandmont, J. M. and G. Laroque (1986) "Stability of Cycles and Expectations," *Journal of Economic Theory*, 40, 138-151.

[18] Grandmont, J. M. and G. Laroque (1990) "Economic Dynamics with Learning : Some Instability Examples," in *Equilibrium Theory and Applications*, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics, ed. by W. A. Barnett et al. Cambridge : Cambridge University Press, 247-273.

[19] Guesnerie, R. and M. Woodford (1991), "Stability of Cycles with Adaptive Learning Rules," in *Equilibrium Theory and Applications*, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics, ed. by W. A. Barnett et al. Cambridge : Cambridge University Press, 111-134.

[20] Kaizoji, T. (1994), "Expecations, Learning Dynamics, and the Stabilization Policy in an Overlapping Generations Models," mimeo, International Christian University, March.

[21] Lucas, Jr., R. E. (1978) "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429-1445.

[22] Lucas, Jr., R. E. (1986) "Adaptive Behavior and Economic Theory," in *Rational Choice : The Contrast between Economics and Psychology*, ed. by R. Hogarth and M. Reder, University of Chicago Press, Chicago.

[23] Marcet, A. and T. J. Sergent (1988), "The Fate of Systems with Adaptive Expectations," *American Economic Review*, 78, 168-172.

[24] Marcet, A. and T. J. Sergent (1989), "Convergence of Least Squares Learning Mechanisms in Self-Referential Stochastic Models," *Journal of Economic Theory*, 48, 337-368.

[25] K. Tarui (1991), "Stability of Fixed Points of Discrete Dynamics System $F(x_{t+1}, x_t) = 0$ II." *IECEJ Technical Report* CAS91-97, 17-22 (in Japanese).

[26] Woodford, M. (1990) "Learning to Believe in Sunspots," *Econometrica*, 58, 277-307.

Figure 1

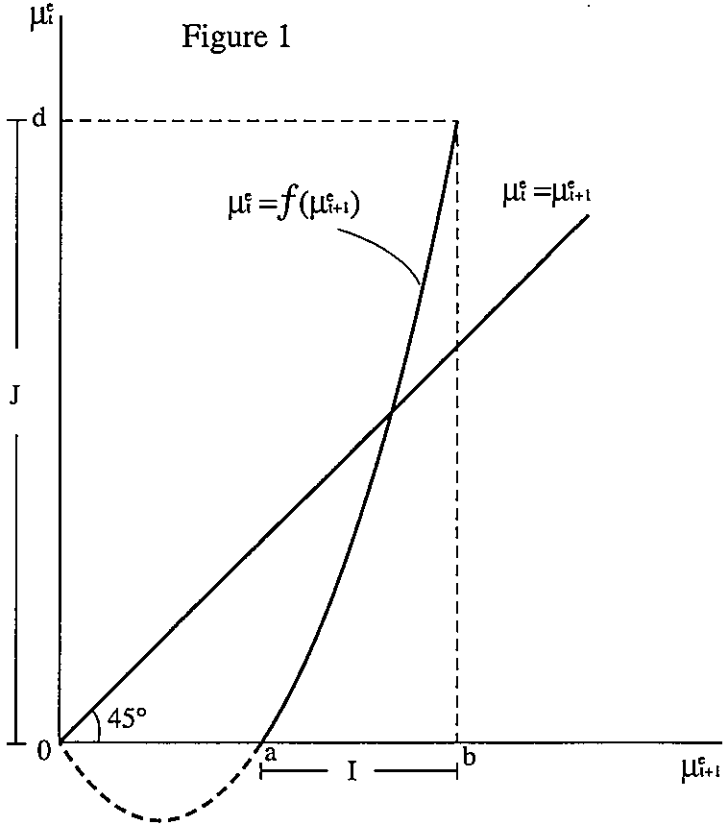


Figure 2: The inverse relation of f

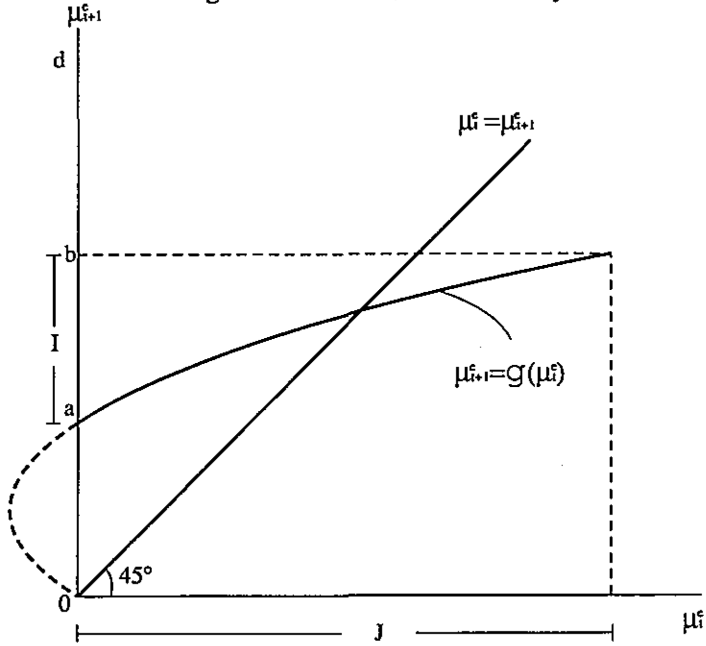
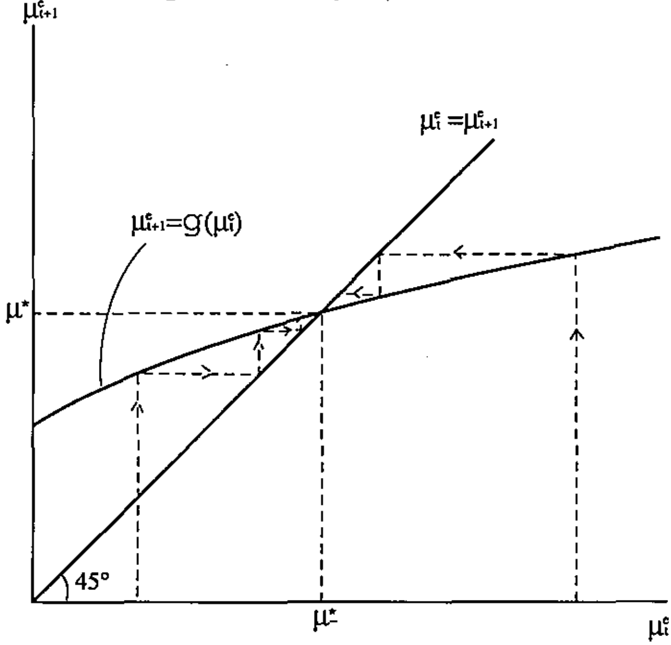


Figure 3: Stability of μ^*



貨幣を含む効用関数モデルにおける適応的学習ルールと完全予見均衡の安定性

海蔵寺大成

最近，様々な経済モデルからカオスが発生することが報告されている。福田（1993）は効用関数が貨幣を含む動的最適化モデルからカオスが発生することを示した。

本稿では，福田モデルを用いて，経済主体が将来の価格水準について適応的学習を行う場合，経済時系列のカオスは学習によって安定化し，経済均衡に収束することを証明する。

本稿の結果は，経済主体が，適切な学習を行えば，長期的には経済変動は安定化することを示している。