A Note on Testing Warrant Pricing Models: An Entropy Based Approach*

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1. Introduction

Pricing options and related derivative instruments such as warrants is a difficult task as informational efficiency is not always the norm. This issue of informational efficiency becomes particularly problematic in emerging markets or in new markets with less liquidity. In this paper we provide an alternative approach to pricing options based on a maximum entropy distribution and apply it to a warrant issued by Thai Farmers Bank (TFB). The maximum entropy distribution (MED) is derived by using the historical return series for TFB stock, the underlying asset, as well as the return series for the Thai stock index. We assume the index provides "information" useful in pricing the TFB warrant. By using the index we are also implicitly using information on the non-diversifiable risk or beta since the correlation between the two series is non-zero.

2. The Entropy Maximization Problem

We assume that at time t, the TFB stock price $SP(t)$ and the Thai index price $IP(t)$, are defined for the time horizon $[0,T]$ where $T > 0$ and are defined on a complete probability space (Ω, F, P). Then the price processes of the TFB stock and the index are stochastic processes defined as $SP(t)$ and $IP(t)$, respectively. If we only consider the first two moments of the random variables, they become elements of $L^2(P)$. $SP(t)$ and $IP(t)$ are assumed to follow Geometric Brownian motion with drift. The discrete version is given in equations (1a) and (1b).

$$\frac{\Delta SP(t)}{SP(t)} = \mu_{SP}\Delta t + \sigma_{SP}\Delta W_{SP} \quad \text{(1a)}$$

$$\frac{\Delta IP(t)}{IP(t)} = \mu_{IP}\Delta t + \sigma_{IP}\Delta W_{IP} \quad \text{(1b)}$$

where μ_{SP} and μ_{IP} are the drift terms, σ_{SP} and σ_{IP} are the volatility terms, and $\Delta W_{SP}(t) = \varepsilon_{SP}\sqrt{\Delta t}$ and $\Delta W_{IP}(t) = \varepsilon_{IP}\sqrt{\Delta t}$. ε_{SP}, ε_{IP} are joint normally distributed with correlation γ.

Let us consider the joint distribution of the discrete random variables $SP(t)$ and $IP(t)$. We define the joint probability distribution function as $P[SP_i = sp_i, IP_j = ip_j] = p_i \cdot q_j$ for $i = 1,......, k$ and $j = 1,......,\ell$. The marginal distributions are given as $P_{SP}[SP_i = sp_i] = p_i$ for $i = 1,......, k$ and $P_{IP}[IP_j = ip_j] = q_j$ for $j = 1,......,\ell$. If we define the first and second moment of the joint distribution function as m_1 and m_2, respectively, and the correlation as $\hat{\gamma}$, the maximum entropy probability distribution of this function is the solution to the following problem (A1).[1]

Problem (A1)

$$
\begin{aligned}
& Max \quad H = -\sum_{i=1}^{k}\sum_{j=1}^{\ell} P(i,j)\log(P(i,j)) \\
& s.t. \\
& \sum_{i=1}^{k}\sum_{j=1}^{\ell} P(i,j) = 1 \\
& \sum_{i=1}^{k}\sum_{j=1}^{\ell} P(i,j) r_j = m_1 \\
& \sum_{i=1}^{k}\sum_{j=1}^{\ell} P(i,j) r_j^2 = m_2 \\
& \sum_{i=1}^{k}\sum_{j=1}^{\ell} P(i,j) r_j f_i = \hat{\gamma} \\
& P(i,j) \geq 0 \quad i = 1,.....,k, \quad for\ j = 1,....,\ell
\end{aligned}
$$

where r_j are the realized discrete values of a random variable which in this case is the index return (for $j=1,......,\ell$). The realized value for the underlying log relative TFB stock return is f_i (for i = 1,......, k). m_1 is the first moment for the TFB returns, m_2 is the second moment for the TFB returns, and $\hat{\gamma}$ is the correlation between TFB and index returns. Solving provides us with the probability, p_i, for i = 1,......, k

As shown in Rajasekera and Yamada (2001), this problem can be solved efficiently when put into a Geometric Programming formulation (GP). In order to facilitate this reformulation, we need to make the following change of variables.

$\delta_{(i-1)\ell+j} = P(i,j)$ for $j=1,......,\ell$ $i=1,......,k$ and $n=k\ell$

This variable conversion allows us to convert the original bivariate problem into a univariate problem. Under this new notation, the original problem (A1) can be reformulated as (D1).

Problem (D1)

$$Max \quad H=\prod_{i=1}^{n}\left(\frac{1}{\delta_i}\right)^{\delta_i}$$

$$s.t.$$

$$\sum_{i=1}^{n}\delta_i=1$$

$$\Gamma^T\delta=0$$

$$\delta_i\geq 0, \quad i=1,.....,n$$

where the matrix Γ^T is defined as the following.

$$\Gamma^T = \begin{bmatrix} r_1 - m_1 & \cdots & r_\ell - m_1 & | & \cdots & | & \cdots & | & r_1 - m_1 & \cdots & r_\ell - m_1 \\ r_1^2 - m_2 & \cdots & r_\ell^2 - m_2 & | & \cdots & | & \cdots & | & r_1^2 - m_2 & \cdots & r_\ell^2 - m_2 \\ f_1 r_1 - \hat{\gamma} & \cdots & f_1 r_\ell - \hat{\gamma} & | & \cdots & | & \cdots & | & f_1 r_1 - \hat{\gamma} & \cdots & f_1 r_\ell - \hat{\gamma} \\ 1 - p_1 & \cdots & 1 - p_1 & | & \cdots & | & \cdots & | & -p_1 & \cdots & -p_1 \\ -p_2 & \cdots & -p_2 & | & \cdots & | & \cdots & | & -p_2 & \cdots & -p_2 \\ \vdots & \ddots & \vdots & | & \vdots & | & \vdots & | & \vdots & \ddots & \vdots \\ 1 - p_{k-1} & \cdots & 1 - p_{k-1} & | & \cdots & | & \cdots & | & 1 - p_{k-1} & \cdots & 1 - p_{k-1} \end{bmatrix}$$

Note that a redundant constraint is created when we apply the law of total probability constraint to the problem, i.e. $\sum_{i=1}^{k} p_i = 1$. This constraint is eliminated from the matrix Γ^T, thus the matrix is a full rank matrix. The matrix Γ^T has 3 + (k–1) rows and $n = k\ell$ columns. This type of optimization problem is called a geometric program, and its Dual is well known. The Dual problem of (D1) is given as the following problem (P1).

Problem (P1)

$$\min \quad g(t) = \sum_{i-1}^{n} t_1^{a_{i1}} t_2^{a_{i2}} \cdots\cdots t_m^{a_{im}}$$
$$s.t.$$
$$t_j > 0, \quad j = 1, \ldots\ldots, m$$

where m =number of rows in M^T = 3 + (k–1) = (k+2) and a_{ij} = (j, i) th element of Γ^T, for i = 1,......, n and j = 1,......, m.

By making the following change of variable, $\exp(\lambda_j)=t_j$, j = 1,......, m, we can convert problem (P1) into an unconstrained version. This significantly improves the computational complexity of the problem. Let us define this unconstrained problem as problem (P2) given below

Problem (P2)

$$\min G(\lambda)=\sum_{i=1}^{n}\exp\left[\sum_{j=1}^{m}\lambda_j a_{ij}\right]$$

$$s.t.$$

$$\lambda_j,\quad j=1,\ldots,m\quad \textit{unconstrained}$$

Because the objective function of problem (P2) is a convex function, the first duality theorem of geometric programming allows us to obtain the solution for problem (A1) from the solution of problem (D1) whose solution also can be obtained efficiently by the solution to problem (P2). It turns out that if we define the solution for problem (P2) as λ_j^*, the solution for problem (D1) δ_j^* can be obtained by the following formula.

$$\delta_i^*=\frac{\exp\left[\sum_{j=1}^{m}\lambda_j^* a_{ij}\right]}{\sum_{i=1}^{n}\exp\left[\sum_{j=1}^{m}\lambda_j^* a_{ij}\right]},\quad i=1,\ldots\ldots,n$$

As a result, we are able to obtain the conditional distribution of the joint probability distribution function of the stock index returns and the underlying stock price returns as

$$P(i\mid j)=P[SP_i=sp_i\mid IP_j=ip_j]=\frac{P[SP_i=sp_i,IP_j=ip_j]}{P[IP_j=ip_j]}\quad \text{for } i=1,\ldots\ldots,k\ \ j=1,\ldots\ldots,\ell\,.$$

Consequently, $P(i\mid j)=\dfrac{P(i,j)}{\sum_{j=1}^{\ell}P(i,j)}=\dfrac{\delta_{(i-1)\ell+1}}{\sum_{j=1}^{\ell}\delta_{(i-1)\ell+j}}$ for $i=1,\ldots\ldots,k\ \ j=1,\ldots\ldots,\ell$

In this paper we will compare how this distribution performs relative to the historical probability distribution. Performance is measured against our benchmark distribution, the implied probability distribution of the underlying TFB stock price returns. The implied probability distribution is derived by employing returns implied from the Black-Scholes based warrant pricing model.

3 . Empirical Analysis

We examine a call warrant issued by Thai Farmers Bank in the Thai stock market during the mid-1990s. The sample coincides with the early stages of the warrant market in Thailand and thus allows us test to the effectiveness of the Entropy based pricing model for emerging and less liquid markets. Daily data is gathered for the Thai Farmers Bank warrant expiring September 30, 1999. We use daily data from January 1995 to October 1995.

The implied price series is derived by applying the warrant pricing model developed by Lauterbach, Schultz (1990).

$$W=(\frac{N}{N/\theta+M})[(SP-\sum_{i}e^{-rq_i}D_i+\frac{M}{N}W)N(d_1)-e^{-r(T-t)}XN(d_s)] \quad (2a)$$

$$d_1=\frac{\ln\left(\frac{SP-\sum_{i}e^{-rq_i}D_i+(M/N)W}{X}\right)+r(T-t)}{\sigma\sqrt{T-t}}+\frac{\sigma\sqrt{T-t}}{2} \quad (2b)$$

$$d_2=d_1-\sigma\sqrt{T-t} \quad (2c)$$

where W is the warrant price, SP is the underlying stock price, X is the exercise price, N is the number of outstanding shares of stock, M is the number of warrants, θ is the number of shares that can be purchased with each warrant, r is the risk-free interest rate, $T-t$ is the time until expiration, σ is the standard deviation of the return of $SP+(M/N)W$ per unit time, $N(d)$ is the cumulative normal distribution function evaluated at d, q_i is the time until the ith dividend is paid, and D_i is the Bhat amount (per share) of the ith dividend.

Firstly, the implied volatility is obtained by iteratively solving equations (2a)-(2c) given the market warrant price. The implied volatilities are then used as forecasts to obtain the implied underlying TFB stock price for the following day again using equations (2a)-(2c) in an iterative exercise. An implied return series is then calculated by using this implied stock price series. Shastri and Sirodom (1995) show that the Cox square root model performs marginally better than the above modified Black-Scholes model. Yet, they conclude that the difference may not be economically significant. For the purpose of this paper, we will assume the Black-Scholes model is correct to the first order.

Table 1: Mean, Standard Deviation, and Correlation of Return Series

Panel A: Mean of Return Series

Date	TFB Stock Return	Implied Stock Return	Thai Index Return
January 23	-0.02	-0.02	-0.23
March 7	-0.02	-0.02	0.04
April 24	-0.03	-0.03	-0.11
June 8	0.34	0.34	0.59
July 21	-0.02	-0.02	-0.03
September 4	-0.20	-0.20	-0.21

Panel B: Standard Deviation and Correlation of Return Series

Date	TFB Stock Return	Implied Stock Return	Thai Index Return	Correlation
January 23	23.92	29.03	25.52	0.826
March 7	24.15	27.90	23.47	0.874
April 24	20.70	27.81	20.80	0.706
June 8	25.70	19.41	23.35	0.842
July 21	21.29	23.05	19.51	0.822
September 4	22.17	30.31	15.93	0.845

Mean and Standard Deviation are in percentage and calculated using thirty non-overlapping trading days. Date indicates the beginning day for the sample. TFB (Thai Farmers Bank) and index returns are obtained using daily closing prices. The implied return is obtained using prices implied from the warrant pricing model. The correlation is between the TFB stock return and Thai index return.

The mean (first moment) and standard deviation (second moment) is estimated for the historical underlying stock price (TFB) return series and Thai index return series for six non-overlapping samples 30 trading days (Table 1). The mean, standard deviation, and correlation between the TFB returns and index returns are used as inputs to derive the maximum entropy distribution (MED). The mean and standard deviation for the return on the implied TFB price series is estimated over the corresponding sample in order to obtain the distribution implied by the warrant prices and warrant pricing model. This implied return distribution (IMP) will serve as a benchmark for comparison.

Table 2: Mean Squared Error of Distributions

Month	MSE for MED vs IMP	MSE for HR vs IMP
January 23	0.0301	0.1306
March 7	0.0311	0.0741
April 24	0.1057	0.3415
June 8	0.4448	0.3421
July 21	0.3848	0.0281
September 4	0.0275	0.3451

MSE is mean squared error and is in percentage. MED is the maximum entropy distribution. HR is the distribution derived from 30 days of historical returns. IMP is the distribution derived from return series implied from the warrant prices and warrant pricing model. All distributions are compared over the same non-overlapping 30 trading day sample. Here we assume the IMP distribution as the benchmark for comparison.

The MED is derived by using the 30 day historical underlying TFB return distribution and the 30 day historical stock index return distribution. This is compared with the implied return distribution derived from warrant prices and the warrant model (IMP). The mean squared error is calculated to gauge the difference between the two empirical distributions. We also obtain the mean squared error between the historical underlying asset return distribution (HR) relative to the distribution implied from the warrant prices (IMP). The mean squared error for each non-overlapping period is summarized in Table2. We find the MSE for the MED does not perform any worse than the 30 day historical return distribution. The MED does not perform well in June and July, however, it should be noted that MED and HR both performed relatively poorly in June. Hence, we provide preliminary evidence that a MED based pricing model could be effective in pricing options and related derivative instruments.

Notes

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(1) In depth discussion and overview of entropy optimization is found in Fang, Tsao, and Rajasekera (1997). Applications of entropy optimization to option pricing include Buchen and Kelly (1996), and Stutzer (2000).

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ワラント価格付けモデルに関する一考察：情報エントロピー最適化による評価法

〈要　約〉

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オプションやワラントといった金融派生商品の価格付けは、市場における情報の伝達効率が不十分であることから非常に難しい問題となる。とくに、商品が十分流動的でないエマージング市場において、価格付けは深刻な問題である。本研究では、こうした問題を解決するために、情報エントロピー最適化を利用した評価方法を使用して、Thai Farmer's Bank 及び株価インデックスに適応した場合の有効性を実証する。